ANALYZING INVESTMENT RETURN OF ASSET PORTFOLIOS WITH MULTIVARIATE ORNSTEIN-UHLENBECK PROCESSES

by

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Abstract

The investment return rates of an asset portfolio can be fitted and analyzed by one univariate Ornstein-Uhlenbeck (O-U) process (global model), several univariate O-U processes (univariate model) or one multivariate O-U process (multivariate model). The expected values, variances and covariance of the instantaneous and accumulated return rates of different asset portfolios are calculated from the three models and compared. Furthermore, we price for annuity products, optimize asset allocation strategy and compare the results. The multivariate model is the most comprehensive and complete of the three models in term of fully capturing the correlation among the assets in a single portfolio.

Keywords: Asset Allocation Strategy; Asset Portfolio; Investment Return Rate; Multivariate Ornstein-Uhlenbeck (O-U) Process

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Contents

Approval ii

Abstract iii

Acknowledgments iv

Contents v

List of Tables vii

List of Figures viii

1 Introduction 1

2 Review of the AR(1) and O-U Processes 4

2.1 One-Dimensional AR(1) Process 5
2.2 One-dimensional Ornstein-Uhlenbeck process 5
2.2.1 Ornstein-Uhlenbeck velocity process 5
2.2.2 Ornstein-Uhlenbeck position process 6
2.3 Converting a one-dimensional AR(1) process to an O-U process 7
2.4 Multivariate AR(1) process 8
2.5 Multivariate O-U process 9
2.5.1 Multivariate O-U Velocity process 9
2.5.2 Multivariate O-U position process 9
2.6 Converting a Multivariate AR(1) Process to an O-U Process 10

3 Investment Models **13**
3.1 Univariate Model 14
3.2 Global Model 16
3.3 Multivariate Model 16
3.3.1 Estimation of the Model Parameters 17
3.3.2 Model Calculation 21

4 Fitting and Comparing Investment Models **29**
4.1 Fitting the Models 29
4.1.1 Testing the Models 31
4.1.2 Fitting the Univariate Model 33
4.1.3 Fitting the Multivariate Model 36
4.1.4 Fitting the Global Model 39
4.2 Comparing the Models 41
4.2.1 Comparing the Instantaneous Return Rates 41
4.2.2 Comparing the Accumulated Return Rates 48

5 Applications **53**
5.1 Annuity Pricing 53
5.1.1 Pricing n-Year Certain Annuity-Immediate Products 54
5.1.2 Pricing Whole-Life Annuity Products 57
5.2 Optimal Asset Allocation Strategy 61

6 Conclusions **67**
6.1 Comparing the Models 68
6.2 Annuity Pricing and Asset Allocation Optimization 69
6.3 Future Work 69

Bibliography **71**

List of Tables

4.1 Comparison of Daily Method and Annual Method 33

5.1 Mean, Standard Deviation and Coefficient of Variation of Z of an n-Year Certain Annuity-Immediate 55

5.2 Asset Portfolios with Minimum and Maximum E(Z) and Var(Z), Starting with Rate of Return in 2007 65

List of Figures

4.1 The Historical Return Rates of Three Assets 30

4.2 Autocorrelation Function and Partial Autocorrelation Function of the Daily Return Rates 32

4.3 Autocorrelation Function and Partial Autocorrelation Function of the Annual Return Rates 34

4.4 $E(X_t|X_0)$ for Each Asset 42

4.5 $V(X_t|X_0)$ for Each Asset 43

4.6 $E(X_t|X_0)$ for Each Portfolio, Starting in 2007, Using Annual Return Rates . 46

4.7 $Var(X_t|X_0)$ for Each Portfolio, Starting in 2007, Using Annual Return Rates 47

4.8 $E(Y_t|X_0)$ for Each Portfolio, Starting in 2007, Using Annual Return Rates . . 50

4.9 $Var(Y_t|X_0)$ for Each Portfolio, Starting in 2007, Using Annual Return Rates 51

4.10 $Var(Y_t|X_0)$ for Each Portfolio for the First Forty Years, Starting in 2007, Using Annual Return Rates 52

5.1 E(Z) for a Whole-Life Annuity 58

5.2 Var(Z) for a Whole-Life Annuity 59

5.3 CV(Z) for a Whole-Life Annuity 60

5.4 E(Z) for a Whole-Life Annuity Sold to Male, Age 65, Starting with Rate of Return in 2007 63

5.5 E(Z) for a Whole-Life Annuity Sold to Male, Age 65, Starting with High Return Rate 64

5.6 Var(Z) for a Whole-Life Annuity Sold to Male, Age 65, Starting with Rate of Return in 2007 66

Chapter 1

Introduction

In actuarial science and finance, the present value of future cash flows is obtained by discounting the cash flows with appropriate discount factors which are reciprocals of the accumulated rate of return. Therefore, how to deal with the investment risk caused by the random nature of the rate of return has been the subject of many publications. Because the investment risk normally cannot be diversified by selling a large number of policies, it could be more important than the mortality risk.

Pollard (1971) allows the rate of return as well as the age at death to vary stochastically and determines premium loadings for non-profit assurances. By modeling annual rates of return as white noise (identically and independently distributed), Boyle (1976) derives the first three moments of interest discount functions and further applied them to traditional actuarial life contingency functions. Following that, Waters (1978) uses an autoregressive model of order one for rates of return, and finds the moments of the present value of the benefit payable under certain types of assurance policies.

Panjer and Bellhouse (1980, 1981) develop a more general theory including continuous and discrete models. They look at moments of assurance and annuity functions when the rates of return are assumed to follow first and second order autoregressive models. Stationary processes are used in their first paper, while conditional processes were used in their second paper.

Giaccotto (1986) develops an algorithm for evaluating actuarial functions when the rates of return are assumed to follow an ARIMA (p, 0, q) or an ARIMA (p, 1, q) process. Dhaene (1989) develops a more efficient method for computing moments of insurance functions when the rates of return are assumed to follow an ARIMA (p,d,q) process.

Beekman and Fuelling (1990, 1991)) derive the first two moments of life annuity functions by modeling the accumulation function of the rates of return with an O-U or Wiener process.

Parker (1993, 1994a, 1994b) provides methods for calculating the first three moments of the present value of future cash flows (P) for a portfolio of insurances or annuities when the rates of return are assumed to be white noise, an O-U process or a Wiener process. Furthermore, he suggests a useful method for approximating the cumulative distribution function of P. In Parker (1995), the first three moments of annuity functions are calculated when the rates of return are modeled by a second order stochastic differential equation.

In all these publications, the authors focus on the rates of return of one asset. In these cases, one univariate process is sufficient for modeling the rates of return. For a portfolio of assets, not only do the rates of return of each asset have their own process, but also the rates of returns of different assets are correlated. Both a single univariate process and several separate univariate processes are inappropriate for describing the rates of return of the entire asset portfolio. Therefore, a multivariate process is essential in evaluating the rates of return of the total asset portfolio. However, the multivariate process is much more complicated.

Wan (2010) models the return rates of asset portfolio as a multivariate O-U process. The aim of this project is to find a numerical method to calculate the first two moments of certain types of assurance and annuity functions when the rates of return are modeled by a multivariate process. With the moments, we can find the expected value and the variance of the net single premium of those products. The results can be compared with those calculated from one or several univariate processes.

In Chapter 2, properties and related formulas of AR(1) and O-U processes and the conversion from an AR(1) to an O-U process is reviewed. Chapter 3 displays the calculations under three investment models for the expected values, variances and covariance of the instantaneous rates of return and accumulation function of the rates of return of asset portfolios. In Chapter 4, we fit three investment models into historical data and compare the results. Chapter 5 discusses some practical applications of the three investment models. We do some annuity pricing and asset allocation strategy optimization with the three models in this chapter. Finally, Chapter 6 presents the conclusions and suggestions for future work.

Chapter 2

Review of the AR(1) and O-U Processes

In this project, we use the Vasicek model to estimate future rates of investment return of the assets. The Vasicek model (Vasicek (1977)) is one of the earliest non-arbitrage interest rate models. It remains one of the most common models in financial and insurance industries to model investment return with some obvious advantages. Vasicek's model is a continuous interest rate model that manages to capture "mean reversion", an essential characteristic of the investment return rate. Additionally, Vasicek's model is a "one-factor" model that provides explicit and simple formulae for calculating yield curves and pricing derivatives. In the Vasicek model, the interest rate is assumed to be driven by a well-known Ornstein-Uhlenbeck (O-U) process, which is a Gaussian process with a bounded variance and admits a stationary probability distribution under certain conditions on the parameter α. The first-order autoregressive process (AR(1)) is frequently proposed for analyzing discrete-time series and can be considered as a discrete time analogue of the continuous O-U process. In reality, all historical data on investment return rates are collected at a certain frequency in a discrete-time frame. Therefore, an AR(1) process is first fitted the data. The AR(1) process is then converted to an equivalent O-U process for studying the investment rate in different time intervals. In the following sections of this chapter, basic properties of the AR(1) and O-U processes will be reviewed. The formulae in this chapter are coming from Pandit and Wu (1983) and Wan (2010).

2.1 One-Dimensional AR(1) Process

An AR(1) process is a stochastic process of the form

$$X_t - \mu = \phi(X_{t-1} - \mu) + a_t, \quad t = 1, 2, \dots \tag{2.1}$$

where μ and ϕ are constants, a_t is a random error term. Normally, μ is the mean of the observations. We assume the a_t's, are independent and follow identical normal distributions, i.e. $a_t \sim N(0, \sigma_a^2)$. If $\phi = 1$, the process is a random walk; if $\phi = 0$, X_t is white noise. If $0 < \phi < 1$, the process is said to be mean-reverting and stationary, which can be converted to a continuous O-U process. All the following properties and equations for an AR(1) are derived assuming that $0 < \phi < 1$.

Suppose that the system starts at a given value X_0. For $t = 1, 2 \dots$, we have

$$X_t - \mu = \phi^t(X_0 - \mu) + \sum_{j=0}^{t-1} \phi^j a_{t-j}. \tag{2.2}$$

Therefore,

$$E(X_t|X_0) = \phi^t(X_0 - \mu) + \mu, \tag{2.3}$$

$$Var(X_t|X_0) = \frac{1 - \phi^{2t}}{1 - \phi^2}\sigma_a^2 \tag{2.4}$$

and

$$Cov(X_s, X_t|X_0) = \phi^{|t-s|}\left(\frac{1 - \phi^{2min(t,s)}}{1 - \phi^2}\right)\sigma_a^2. \tag{2.5}$$

When t goes to infinite,

$$\lim_{t\to\infty} E[X_t|X_0] = \mu \tag{2.6}$$

and

$$\lim_{t\to\infty} Var(X_t|X_0) = \frac{\sigma_a^2}{1 - \phi^2}. \tag{2.7}$$

2.2 One-dimensional Ornstein-Uhlenbeck process

2.2.1 Ornstein-Uhlenbeck velocity process

The O-U process is a continuous analogue of the AR(1) process and given by the following stochastic differential equation (SDE)

$$dX_t = -\alpha(X_t - \mu)dt + \sigma dW_t, \quad ,t \geq 0 \tag{2.8}$$

where α, μ and σ are non-negative constants. In the Vasicek model, X_t is the current level of the investment return rate, and μ is the long-term mean of the rate of investment return. Parameter α measures the market force pushing the rate toward its long-term mean and describes how fast the current investment return rate reverts to its long-run norm, while parameter σ determines the instantaneous volatility of the investment return rate. W_t is a standard Brownian motion. Solving SDE(2.8), we get

$$X_t - \mu = e^{-\alpha t}(X_0 - \mu) + \sigma \int_0^t e^{-\alpha(t-s)} dW_s, t \geq 0 \tag{2.9}$$

where the initial value X_0 is the rate of return at time 0. Given the initial value X_0, X_t is a Gaussian process and we have

$$E(X_t|X_0) = e^{-\alpha t}(X_0 - \mu) + \mu, \tag{2.10}$$

$$Var(X_t|X_0) = \frac{1 - e^{-2\alpha t}}{2\alpha}\sigma^2 \tag{2.11}$$

and

$$Cov(X_s, X_t|X_0) = e^{-\alpha(t+s)} \left(\frac{e^{2\alpha min(s,t)} - 1}{2\alpha} \right) \sigma^2. \tag{2.12}$$

When t goes to infinite,

$$\lim_{t\to\infty} E(X_t|X_0) = \mu, \tag{2.13}$$

and

$$\lim_{t\to\infty} Var(X_t|X_0) = \frac{\sigma^2}{2\alpha}. \tag{2.14}$$

2.2.2 Ornstein-Uhlenbeck position process

It is convenient to obtain a closed and simple formula to describe the accumulation function of the instantaneous rate of investment return (Y_t) in the O-U process. The O-U position process, Y_t, is obtained by integrating the process X_t. So

$$Y_t = Y_0 + \int_0^t X_s ds. \tag{2.15}$$

In the financial world, Y_0 is normally 0 and X_0 is a known constant, implying that Y_t is a Gaussian process. Therefore, we get

$$E(Y_t|X_0) = \frac{1-e^{-\alpha t}}{\alpha}(X_0-\mu)+\mu t, \tag{2.16}$$

$$Var(Y_t|X_0) = \frac{\sigma^2}{\alpha^2}t+\frac{\sigma^2}{2\alpha^3}\left(-3+4e^{-\alpha t}-e^{-2\alpha t}\right) \tag{2.17}$$

and

$$Cov(Y_s,Y_t|X_0) = \frac{\sigma^2}{\alpha^2}min(s,t)+\frac{\sigma^2}{2\alpha^3}\left[-2+2e^{-\alpha t}+2e^{-\alpha s}-e^{-\alpha|t-s|}-e^{-\alpha(t+s)}\right]. \tag{2.18}$$

2.3 Converting a one-dimensional AR(1) process to an O-U process

Due to the discreteness of the past observations of the return rate and the great advantage of using a continuous process to study future return rates, conversion from an AR(1) process to an O-U process by the principle of covariance equivalence (Pandit and Wu. (1983)) is an important and necessary step in our project.

Suppose the system is sampled at $0, \Delta, 2\Delta, 3\Delta \ldots$. The conditional covariance between the observations at time $t\Delta$ and k intervals earlier for an O-U process can be obtained from Equation (2.12)

$$Cov(X_{t\Delta},X_{t\Delta-k\Delta}|X_0) = e^{-\alpha k\Delta}(\frac{1-e^{-2\alpha(t\Delta-k\Delta)}}{2\alpha})\sigma^2, \tag{2.19}$$

The autocovariance function of a conditional AR(1) with a time period of k units is obtained from Equation (2.5)

$$Cov(X_t,X_{t-k}|X_0) = \phi^k(\frac{1-\phi^{-2(t-k)}}{1-\phi^2})\sigma^2. \tag{2.20}$$

By matching the time of observation t in Equation (2.20) with time $t\Delta$ in Equation (2.19), the two covariance functions above are assumed to have the same values for any positive integers t and k in the covariance equivalent principle. By matching the coefficients, we have

$$\alpha = \frac{-\ln\phi}{\Delta} \text{ or } \phi = e^{-\alpha\Delta}, \tag{2.21}$$

and

$$\frac{\sigma_a^2}{1-\phi^2} = \frac{\sigma^2}{2\alpha}. \tag{2.22}$$

From Equation (2.21), it can be seen that every uniformly sampled O-U process has an corresponding AR(1) expression. However, not every AR(1) process can be written as a continuous and stationary O-U process. This can happen only when $0 < \phi < 1$. Notethat σ, σ_a, α, and ϕ in Equation (2.22) must be defined for the same time unit.

2.4 Multivariate AR(1) process

Compared to a combination of several univariate AR(1) processes, a multi-dimensional AR(1) process can not only display some autocorrelation within each time series but can also display the correlations that exist among the series. By expressing every component in Equation (2.1) in vector or matrix form, we get the following describing a group of time series as a multivariate AR(1) process

$$\underline{X}_t - \underline{\mu} = \underline{\Phi}(\underline{X}_{t-1} - \underline{\mu}) + \underline{a}_t \quad t = 1, 2, \ldots \tag{2.23}$$

or

$$\begin{bmatrix} X_{1,t} - \mu_1 \\ X_{2,t} - \mu_2 \\ \vdots \\ X_{n,t} - \mu_n \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} & \ldots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \ldots & \phi_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ \phi_{n1} & \phi_{n2} & \ldots & \phi_{nn} \end{bmatrix} \cdot \begin{bmatrix} X_{1,t-1} - \mu_1 \\ X_{2,t-1} - \mu_2 \\ \vdots \\ X_{n,t-1} - \mu_n \end{bmatrix} + \begin{bmatrix} a_{1,t} \\ a_{2,t} \\ \vdots \\ a_{n,t} \end{bmatrix}$$

where $\underline{\Phi}$ is an $n \times n$ constant matrix describing the relationships between each $\underline{X}_{k,t}$ and $\underline{X}_{s,t-1}$ $(t \geq 1)$. Vector$\underline{a}_t$ follows a multivariate normal distribution with mean $\underline{0}$ and covariance matrix $\underline{\Sigma}_a$.

Suppose that the system starts at a given vector $\underline{X}_0$. For $t = 1, 2 \ldots$,

$$\underline{X}_t - \underline{\mu} = \underline{\Phi}^t(\underline{X}_0 - \underline{\mu}) + \sum_{j=0}^{t-1} \underline{\Phi}^j \underline{a}_{t-j}. \tag{2.24}$$

Therefore,

$$E(\underline{X}_t | \underline{X}_0) = \underline{\Phi}^t(\underline{X}_0 - \underline{\mu}) + \underline{\mu}, \tag{2.25}$$

$$Var(\underline{X}_t | \underline{X}_0) = \sum_{i=0}^{t-1} \underline{\Phi}^i \underline{\Sigma}_a (\underline{\Phi}^i)^T \tag{2.26}$$

and

$$Cov(\underline{X}_t, \underline{X}_{t-k} | \underline{X}_0) = \sum_{i=0}^{t-k-1} \underline{\Phi}^{k+i} \underline{\Sigma}_a (\underline{\Phi}^i)^T. \tag{2.27}$$

where $(\underline{\Phi}^i)^T$ denotes the transpose of matrix $\underline{\Phi}^i$.

2.5 Multivariate O-U process

2.5.1 Multivariate O-U Velocity process

For a multidimensional O-U velocity process,

$$d\underline{X}_t = \underline{A}(\underline{X}_t - \underline{\mu})dt + \underline{\sigma} d\underline{W}_t, \quad t \geq 0 \tag{2.28}$$

or

$$d\begin{bmatrix} X_{1,t} \\ X_{2,t} \\ \vdots \\ X_{n,t} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix} \cdot \begin{bmatrix} X_{1,t} - \mu_1 \\ X_{2,t} - \mu_2 \\ \vdots \\ X_{n,t} - \mu_n \end{bmatrix} dt + \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1n} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \sigma_{n1} & \sigma_{n2} & \dots & \sigma_{nn} \end{bmatrix} \cdot d \begin{bmatrix} W_{1,t} \\ W_{2,t} \\ \vdots \\ W_{n,t} \end{bmatrix}$$

where $\underline{A}$ is an $n \times n$ constant matrix which describes how the instantaneous change of each series depends on both its current status and the statuses of other series. Matrix $\underline{\sigma}$ is the $n \times n$ diffusion matrix and $\underline{W}_t$ is a vector of n independent standard Brownian Motions. Given $\underline{X}_0$, the solution of the stochastic differential equation (2.28) is

$$\underline{X}_t - \underline{\mu} = e^{\underline{A}t}(\underline{X}_0 - \underline{\mu}) + \int_0^t e^{\underline{A}(t-s)} \underline{\sigma} d\underline{W}_s \quad t \geq 0. \tag{2.29}$$

The conditional expected value of $\underline{X}_t$, given an initial value $\underline{X}_0$, is

$$E(\underline{X}_t | \underline{X}_0) = e^{\underline{A}t}(\underline{X}_0 - \underline{\mu}) + \underline{\mu}. \tag{2.30}$$

Matrix calculation yields the variance and covariance functions of $\underline{X}_s$ and $\underline{X}_t$

$$Var(\underline{X}_t | \underline{X}_0) = e^{\underline{A}t} \left[\int_0^t (e^{\underline{A}u})^{-1} \underline{\Sigma}_{OU} ((e^{\underline{A}u})^{-1})^T du \right] (e^{\underline{A}t})^T, \tag{2.31}$$

and

$$Cov(\underline{X}_s, \underline{X}_t | \underline{X}_0) = e^{\underline{A}s} \left[\int_0^{min(s,t)} (e^{\underline{A}u})^{-1} \underline{\Sigma}_{OU} ((e^{\underline{A}u})^{-1})^T du \right] (e^{\underline{A}t})^T \tag{2.32}$$

where $\underline{\Sigma}_{OU} = \underline{\sigma} \cdot \underline{\sigma}^T$.

2.5.2 Multivariate O-U position process

The multidimensional O-U position process, $\underline{Y}_t$, is obtained by integrating the process, $\underline{X}_t$. So

$$\underline{Y}_t = \underline{Y}_0 + \int_0^t \underline{X}_s ds. \tag{2.33}$$

We could also consider the system of SDE

$$d\begin{bmatrix} \underline{X}_t \\ \underline{Y}_t \end{bmatrix} = \underline{B}\cdot\begin{bmatrix} \underline{X}_t-\underline{\mu} \\ \underline{Y}_t-\underline{\mu}t \end{bmatrix} dt + \underline{\sigma}_Y d\begin{bmatrix} \underline{W}_t \\ \underline{W}_t \end{bmatrix} \tag{2.34}$$

where $\underline{B} = \begin{bmatrix} \underline{A} & \underline{0} \\ \underline{E} & \underline{0} \end{bmatrix}$, $\underline{E}$ is an n-dimensional identity matrix and $\underline{\sigma}_Y = \begin{bmatrix} \underline{\sigma} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}$. By solving the SDE (2.34), we get

$$\begin{bmatrix} \underline{X}_t-\underline{\mu} \\ \underline{Y}_t-\underline{\mu}t \end{bmatrix} = e^{\underline{B}t}\begin{bmatrix} \underline{X}_0-\underline{\mu} \\ \underline{Y}_0 \end{bmatrix} + \int_0^t e^{\underline{B}(t-s)}\underline{\sigma}_Y d\underline{W}s. \tag{2.35}$$

In the financial world, normally $\underline{Y}_0$ is 0.

2.6 Converting a Multivariate AR(1) Process to an O-U Process

In practice, we fit an AR(1) process to the observations and obtain parameters $\underline{\Phi}$ and $\underline{\Sigma}_a$ first. To covert an AR(1) to an O-U process, the covariance equivalence principle is applied to yield the parameters $\underline{A}$, $\underline{\Sigma}_{OU}$ of the corresponding O-U process. To satisfy the principle of covariance equivalence, the first and second moments of the AR(1) process and the O-U process must be equival for any integer unit time t. Therefore, from Equations (2.25) and (2.30), we have

$$E(\underline{X}_t|\underline{X}_0) = \underline{\Phi}^t(\underline{X}_0-\underline{\mu}) + \underline{\mu} = e^{\underline{A}t}(\underline{X}_0-\underline{\mu}) + \underline{\mu}, \tag{2.36}$$

which implies that

$$e^{\underline{A}} = \underline{\Phi}. \tag{2.37}$$

Since $\underline{X}_t$ is a Gaussian process, the covariance function between these two processes will be equal at any time points if they are equal at one time point. Therefore, we can solve for $\underline{\Sigma}_{OU}$ by matching the conditional variance of $\underline{X}_1$ for the multivariate AR(1) process and the one for the O-U process. Letting $t = s = 1$ and $k = 0$ in Equations (2.27) and (2.31), we then have the $Var(\underline{X}_1|\underline{X}_0)$ for both the AR(1) process and the O-U process

$$Var(\underline{X}_1|\underline{X}_0) = \underline{\Sigma}_a = e^{\underline{A}}\left[\int_0^1 (e^{\underline{A}u})^{-1}\underline{\Sigma}_{OU}((e^{\underline{A}u})^{-1})^T du\right](e^{\underline{A}})^T. \tag{2.38}$$

By matching all the n^2 matrix elements of the two sides of the equation, the $n \times n$ matrix of $\underline{\Sigma}_{OU}$ can be solved. Since $\underline{\Sigma}_{OU} = \underline{\sigma}\cdot\underline{\sigma}^T$, there are multiple solutions for $\underline{\sigma}$ if a solution exists. To simplify later calculations, we define $\underline{\sigma}$ as a lower triangular matrix such that $\underline{\sigma}$ can be determined from $\underline{\Sigma}_{OU}$ through Cholesky decomposition.

Although the mathematical expressions for the multivariate process are similar to those for the univariate process, the computational problem is much more challenging when the number of series in vector $\underline{X}_t$ is increased. For example, obtaining $\underline{A}$ from $\underline{\Phi}$ by Equation (2.37) and the explicit expression for $e^{\underline{A}t}$ at any continuous t are more difficult when $\underline{A}$ is expanded from one dimension to multiple dimensions. To determine $\underline{A}$, we first find the eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and the corresponding eigenvectors $([v_{11}, v_{21}, \ldots, 1]^T$, $[v_{12}, v_{22}, \ldots, 1]^T$, $\ldots$, $[v_{1n}, v_{2n}, \ldots, 1]^T)$ of the $n \times n$ matrix $\underline{\Phi}$. Since $e^{\underline{A}} = \underline{\Phi}$, the eigenvalues of $\underline{A}$ are $\mu_1, \mu_2, ..., \mu_n$ with $\mu_i = \ln(\lambda_i)$ for $i = 1, 2, \ldots, n$, and the corresponding eigenvectors are the same as $\underline{\Phi}$. The definitions of the eigenvalue and eigenvector can generate a system of linear equations

$$\mu \underline{v} = \underline{A v} \tag{2.39}$$

where μ is one of the eigenvalues of $\underline{A}$ and $\underline{v}$ is the corresponding eigenvector. $\underline{A}$ is then expressed as

$$\underline{A} = \underline{V}\,\underline{M}\,\underline{V}^{-1} \tag{2.40}$$

where $\underline{M}$ is a diagonal matrix made by eigenvalues of $\underline{A}$ and each column in the $n \times n$ matrix $\underline{V}$ is the corresponding eigenvector. Additionally, to get a stationary O-U process, all the eigenvalues of $\underline{\Phi}$ must lie in the interval between 0 and 1.

The explicit expression of $e^{\underline{A}t}$ is also obtained from the eigenvalues and the eigenvectors of $\underline{A}$. If we have the eigenvalues $(\mu_1, \mu_2, \ldots, \mu_n)$ and the eigenvector matrix $(\underline{V})$ of $\underline{A}$, then

$$E(\underline{X}_t|\underline{X}_0) = e^{\underline{A}t}\underline{X}_0 = \begin{bmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 e^{\mu_1 t} \\ c_2 e^{\mu_2 t} \\ \vdots \\ c_n e^{\mu_n t} \end{bmatrix} \tag{2.41}$$

where $c_1, c_2, \ldots, c_n$ are constants. When $t = 0$, we have

$$E(\underline{X}_t|\underline{X}_0) = \underline{X}_0 = \begin{bmatrix} v_{11} & v_{12} & \ldots & v_{1n} \\ v_{21} & v_{22} & \ldots & v_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & 1 & \ldots & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}. \tag{2.42}$$

From Equation (2.42), we have $\underline{c} = \underline{V}^{-1}\underline{X}_0$. By plugging $\underline{c}$ into Equation (2.41) and matching the coefficients of $\underline{X}_0$ for all rows on both sides of the matrices, we obtain an explicit expression of $e^{\underline{A}t}$. Detailed instructions for solving a 3×3 will be presented in the next chapter.

The procedures involved in solving for matrices $\underline{A}$ and $e^{\underline{A}t}$ demonstrate that there are many more computational complexities in the multi-dimensional O-U process, although it shares similarities with the formulae described for the one-dimensional O-U process. In the next chapter, we will discuss in details how to derive the three-dimensional O-U velocity process from the AR(1), as well as how to develop explicit expressions for the moments of the O-U position process.

Chapter 3

Investment Models

In this chapter, three different models for analyzing a portfolio of various assets are discussed. The first one is a univariate model, which models the rate of return for each one of the assets by a univariate process. The second one, that we call the global model, calculates the total return of the asset portfolio with a preassigned weight on each asset, and then uses a one dimensional stochastic process to model the portfolio's global return. The third one is the multivariate model, which fits a multivariate process to the rates of return of the assets. Of the three, the multivariate model is the most complex and has been studied the least so far. On the other hand, it is the most comprehensive and complete model because it takes into consideration all the correlations between the assets in one portfolio. This chapter will cover in details how to estimate the parameters of a multivariate model. Furthermore, a deterministic method is developed for computing the first and second moments of the accumulation function of the return rate for the portfolio under the multivariate model. The accumulation function of the rate of return is crucial for calculating the present value or future value of the asset portfolio and pricing insurance products. Our approach avoids the time-consuming simulation procedures and provides an efficient way to evaluate the risk of insurance products under this model. In this project, asset portfolios including three different asset components are introduced. In reality, a portfolio with more asset components can be studied by using the same principle and method, although it may present greater computation at challenges.

In this project, each of the assets in the portfolio is set at a certain proportion of the entire portfolio at the start time point ($t = 0$) and maintains at a fixed percentage throughout the investment period. In other words, the assets are rebalanced so frequently that the ratios of their accumulated value to the whole portfolio at any time remain constant. We assume the transaction cost is zero. Therefore, the total investment return rate over any investment time frame t for the whole assets is $\Sigma_{i=1}^{n}\omega_i \int_0^t \delta_{i,s} ds$, where ω_i is the proportion of each asset in the portfolio. Three models are explored to fit and study the rate of return of these frequently rebalanced portfolios.

3.1 Univariate Model

Among these three models, the univariate model is the most straightforward and intuitive. In this model, each of the different assets is estimated as a separate process, and the rate of return of each asset is calculated and evaluated based on these individual models. Then for any time t, the total return rate of the portfolio is calculated by summation of each asset return rate with its corresponding weight in the asset portfolio.

Here we assume that there are three assets in the portfolios and their proportions are fixed. The time series of each asset return rate is modeled by an AR(1) process and transferred to a continuous O-U process as described in Chapter 2. In this way, we obtain three O-U processes to describe three series of return rates of assets

$$X_{i,t} - \mu_i = e^{-\alpha_i t}(X_{i,0} - \mu_i) + \sigma_i \int_0^t e^{-\alpha_i (t-s)} ds, \quad i = 1, 2, 3 \tag{3.1}$$

where $X_{i,t}$ is the instantaneous rate of return of Asset i at time t; $X_{i,0}$ is the instantaneous rate of return of Asset i at initial time 0; α_i and σ_i are the parameters of the one-dimensional O-U process to describe the behavior of the return rate of Asset i.

In the next step, the total rate of return of the portfolio is described as a combination of three asset return rates by their corresponding proportional weights, that is

$$X_{P,t} = p_1 \cdot X_{1,t} + p_2 \cdot X_{2,t} + p_3 \cdot X_{3,t} \tag{3.2}$$

where p_1, p_2, p_3 are the proportions of three assets in the portfolio, and $p_1 + p_2 + p_3 = 1$. $X_{P,t}$ is the instantaneous rate of return of the portfolio.

Furthermore, for any time t, we have

$$\begin{aligned} E(X_{P,t}) &= p_1 \cdot E(X_{1,t}) + p_2 \cdot E(X_{2,t}) + p_3 \cdot E(X_{3,t}) \\ &= p_1 \cdot (e^{-\alpha_1 t} X'_{1,0} + \mu_1) + p_2 \cdot (e^{-\alpha_2 t} X'_{2,0} + \mu_2) + p_3 \cdot (e^{-\alpha_3 t} X'_{3,0} + \mu_3), \quad (3.3) \\ Var(X_{P,t}) &= p_1^2 \cdot Var(X_{1,t}) + p_2^2 \cdot Var(X_{2,t}) + p_3^2 \cdot Var(X_{3,t}) \quad (3.4) \end{aligned}$$

and

$$Cov(X_{P,t}, X_{P,s}) = p_1^2 \cdot Cov(X_{1,t}, X_{1,s}) + p_2^2 \cdot Cov(X_{2,t}, X_{2,s}) + p_3^2 \cdot Cov(X_{3,t}, X_{3,s}) \quad (3.5)$$

where $X'_{n,0} = X_{n,0} - \mu_n$.

In this model, it is assumed that there is no correlation between the rates of return of these three assets. Hence, the covariances between each pair of $X_{t,1}, X_{t,2}$ and $X_{t,3}$ are all zero in Equation 3.5. Because the accumulation function of $\underline{X}_t$, $\underline{Y}_t$, equals $\underline{Y}_0 + \int_0^t \underline{X}_s ds$, we can also derive the formulae for the accumulation function of the return rate ($Y_{P,t}$) for the portfolio in the univariate model

$$Y_{P,t} = p_1 \cdot Y_{1,t} + p_2 \cdot Y_{2,t} + p_3 \cdot Y_{3,t}, \quad (3.6)$$

$$E(Y_{P,t}) = p_1 \cdot E(Y_{1,t}) + p_2 \cdot E(Y_{2,t}) + p_3 \cdot E(Y_{3,t}), \quad (3.7)$$

$$Var(Y_{P,t}) = p_1^2 \cdot Var(Y_{1,t}) + p_2^2 \cdot Var(Y_{2,t}) + p_3^2 \cdot Var(Y_{3,t}) \quad (3.8)$$

and

$$Cov(Y_{P,t}, Y_{P,s}) = p_1^2 \cdot Cov(Y_{1,t}, Y_{1,s}) + p_2^2 \cdot Cov(Y_{2,t}, Y_{2,s}) + p_3^2 \cdot Cov(Y_{3,t}, Y_{3,s}). \quad (3.9)$$

where $Y_{i,t} = Y_{i,0} + \int_0^t X_{i,s} ds$ for i= 1, 2, 3. Explicit expressions for conditional variances and covariances involving $X_{t,n}$ and $Y_{t,n}$ can be obtained from Equations (2.11), (2.12), (2.17) and (2.18) since the instantaneous rate of return for each asset is fitted by a simple one-dimensional O-U process.

The univariate model is easy to understand and simple to calculate, but it has apparent disadvantages. The model completely ignores the correlation between assets. In today's financial world there are plenty of chances that the assets in one portfolio are highly correlated. In those cases, such correlations can have a significant impact on predicting return rates of the portfolio.

3.2 Global Model

Recognizing the limitations associated with a failure to estimate the correlation among assets, a global model is introduced in this project. Instead of creating a separate process for each asset from past observations of its return rate, the return rates of different assets at each time point are combined based on the proportions of the corresponding assets. Then the combined rates are considered as the past return rates of the portfolio from which a one-dimensional stochastic process can be estimated to model the return rate of the total portfolio. In this project, the historical return rate of the whole portfolio is determined by aggregating the rates of return of three asset components with their weights in the portfolio. The historical return rate of the portfolio is used to build a one-dimensional AR(1) process which is then converted to a one-dimensional O-U process. Once having the estimated σ_P and α_P of the O-U process, we have

$$X_{P,t} - \mu_P = e^{-\alpha_P t}(X_{P,0} - \mu_P) + \sigma_P \int_0^t e^{-\alpha(t-s)} dW_s. \tag{3.10}$$

The closed-form formulae for the first and second moments of $X_{P,t}$ and $Y_{P,t}$ can be obtained as described in Chapter 2 since we know $X_{P,t}$ follows an O-U process.

The global model is also simple to understand and calculate. Moreover, unlike the univariate model, it takes into consideration the correlations between the assets. However, the model's simplicity makes it difficult to fully capture all the relationships. It is almost impossible to describe the rates of return of the assets clear in a model with only two parameters.

3.3 Multivariate Model

To fully disclose the dependence among the different assets, a multivariate model is introduced. In this model, a multi-dimensional O-U process are fitted by the rates of return of all the assets. Now the rate of return of each asset is related not only to past observations of that particular asset, but also to past observations of the return rates of other assets. Therefore, the model allows the process to have a more precise estimation of the rates of return of each asset and the asset portfolio. On the other hand, the model has much greater computational complexity than the univariate and global models.

In this project, a three-dimension AR(1) process is applied to model the return rate of the portfolio including three assets. Then, the AR(1) process is converted into an equivalent three-dimensional O-U process. The three-dimensional stochastic O-U process has a 3×3 matrix $\underline{A}$ to describe the correlation among the drifts of the three series and a 3×3 matrix $\underline{\sigma}$ to describe jointly the correlation among the stochastic variations of these series. From the three-dimensional O-U process, we can get the first and second moments of $\underline{X}_t$ to estimate the expected present value and riskiness of the assets. The formulae and methods are basically following Wan(2010). Furthermore, we will display the mathematical derivations of the first and second moments of $\underline{Y}_t$ step by step in the last several paragraphs.

3.3.1 Estimation of the Model Parameters

In the first step, we have a three-dimensional AR(1) based on the past observations of the return rates of each asset to describe the instantaneous rate $\underline{X}_t$:

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{bmatrix} = \begin{bmatrix} \hat{\phi}_{11} & \hat{\phi}_{12} & \hat{\phi}_{13} \\ \hat{\phi}_{21} & \hat{\phi}_{22} & \hat{\phi}_{23} \\ \hat{\phi}_{31} & \hat{\phi}_{32} & \hat{\phi}_{33} \end{bmatrix} \cdot \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \\ X_{3,t-1} \end{bmatrix} + \begin{bmatrix} \hat{a}_{1,t} \\ \hat{a}_{2,t} \\ \hat{a}_{3,t} \end{bmatrix}, \quad t = 1, 2, \dots \tag{3.11}$$

where $\underline{\hat{a}}_t$ follows a multivariate normal distribution with mean $\underline{0}$ and a 3×3 covariance matrix $\underline{\Sigma}_a$. Note that $X_{i,t}$ $(i = 1, 2, 3)$ has been centered at 0 by subtracting each mean from its corresponding series. Given the initial values of $\underline{X}_0$ at time $t = 0$, we further have

$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{bmatrix} = \begin{bmatrix} \hat{\phi}_{11} & \hat{\phi}_{12} & \hat{\phi}_{13} \\ \hat{\phi}_{21} & \hat{\phi}_{22} & \hat{\phi}_{23} \\ \hat{\phi}_{31} & \hat{\phi}_{32} & \hat{\phi}_{33} \end{bmatrix}^t \cdot \begin{bmatrix} X_{1,0} \\ X_{2,0} \\ X_{3,0} \end{bmatrix} + \sum_{j=0}^{t-1} \begin{bmatrix} \hat{\phi}_{11} & \hat{\phi}_{12} & \hat{\phi}_{13} \\ \hat{\phi}_{21} & \hat{\phi}_{22} & \hat{\phi}_{23} \\ \hat{\phi}_{31} & \hat{\phi}_{32} & \hat{\phi}_{33} \end{bmatrix}^j \cdot \begin{bmatrix} \hat{a}_{1,t-j} \\ \hat{a}_{2,t-j} \\ \hat{a}_{3,t-j} \end{bmatrix}. \tag{3.12}$$

In the second step, the parameter $\underline{A}$ of the corresponding three-dimensional O-U process is calculated from the AR(1) process. An explicit expression for $e^{\underline{A}t}$ is also derived in this step. The eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$ and the corresponding eigenvectors ($[v_{11}, v_{21}, 1]^T$, $[v_{12}, v_{22}, 1]^T$, $[v_{13}, v_{23}, 1]^T$) of matrix $\underline{\hat{\Phi}}$ can be solved numerically. From Section (2.6) and Equation (2.37), the eigenvalues of $\underline{A}$ are known as $\mu_1 = \ln(\lambda_1), \mu_2 = \ln(\lambda_2)$ and $\mu_3 = \ln(\lambda_3)$. Additionally, $\underline{A}$ and $\underline{\Phi}$ share the same eigenvectors. Therefore, matrix $\underline{A}$ can be solved from Equation

(2.40):

$$\underline{A} = \underline{V}\,\underline{M}\,\underline{V}^{-1} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \ln(\lambda_1) & 0 & 0 \\ 0 & \ln(\lambda_2) & 0 \\ 0 & 0 & \ln(\lambda_3) \end{bmatrix} \cdot \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ 1 & 1 & 1 \end{bmatrix}^{-1} . \quad (3.13)$$

An explicit expression for $e^{\underline{A}t}$ can also be obtained from the eigenvalues and eigenvectors of $\underline{A}$ through the following steps. From Equation (2.41), the conditional means of the series are

$$E(\underline{X}_t|\underline{X}_0) = e^{\underline{A}t}\underline{X}_0 = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 e^{\mu_1 t} \\ c_2 e^{\mu_2 t} \\ c_3 e^{\mu_3 t} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \lambda_1^t \\ c_2 \lambda_2^t \\ c_3 \lambda_3^t \end{bmatrix} \quad (3.14)$$

where c_1, c_2, c_3 are constants. When $t = 0$, we have

$$\underline{X}_0 = \begin{bmatrix} X_{1,0} \\ X_{2,0} \\ X_{3,0} \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad (3.15)$$

$$\underline{c} = \underline{V}^{-1} \cdot \underline{X}_0. \quad (3.16)$$

Therefore, c_1, c_2, c_3 can be expressed in term of $\underline{X}_0$ and $\underline{V}$

$$c_1 = \frac{(v_{22} - v_{23})X_{1,0} + (v_{13} - v_{12})X_{2,0} + (v_{12}v_{23} - v_{13}v_{22})X_{3,0}}{(v_{11}v_{22} - v_{12}v_{21}) - (v_{11}v_{23} - v_{13}v_{21}) + (v_{12}v_{23} - v_{13}v_{22})}, \quad (3.17)$$

$$c_2 = \frac{(v_{23} - v_{21})X_{1,0} + (v_{11} - v_{13})X_{2,0} + (v_{13}v_{21} - v_{11}v_{23})X_{3,0}}{(v_{11}v_{22} - v_{12}v_{21}) - (v_{11}v_{23} - v_{13}v_{21}) + (v_{12}v_{23} - v_{13}v_{22})}, \quad (3.18)$$

$$c_3 = \frac{(v_{21} - v_{22})X_{1,0} + (v_{12} - v_{11})X_{2,0} + (v_{11}v_{22} - v_{12}v_{21})X_{3,0}}{(v_{11}v_{22} - v_{12}v_{21}) - (v_{11}v_{23} - v_{13}v_{21}) + (v_{12}v_{23} - v_{13}v_{22})}. \quad (3.19)$$

To simplify the expression, we let β denote the denominator of $\underline{c}$. In fact, β is equal to the determinant of the eigenvector matrix $\underline{V}$. By plugging $\underline{c}$ into Equation (3.14) and matching the coefficients of $\underline{X}_0$ for each row on both sides of the matrices, we obtain the following expression for $e^{\underline{A}t}$

$$e^{\underline{A}t} = \begin{bmatrix} \omega_{11}(t,0) & \omega_{12}(t,0) & \omega_{13}(t,0) \\ \omega_{21}(t,0) & \omega_{22}(t,0) & \omega_{23}(t,0) \\ \omega_{31}(t,0) & \omega_{32}(t,0) & \omega_{33}(t,0) \end{bmatrix} \quad (3.20)$$

where $\underline{\omega}(t,s)$ are a series of functions depending on time t and s

$$\begin{aligned}
\omega_{11}(t,s) =& \frac{1}{\beta}[v_{11}(v_{22}-v_{23})\lambda_1^{(t-s)} - v_{12}(v_{21}-v_{23})\lambda_2^{(t-s)} + v_{13}(v_{21}-v_{22})\lambda_3^{(t-s)}],\\
\omega_{21}(t,s) =& \frac{1}{\beta}[v_{21}(v_{22}-v_{23})\lambda_1^{(t-s)} - v_{22}(v_{21}-v_{23})\lambda_2^{(t-s)} + v_{23}(v_{21}-v_{22})\lambda_3^{(t-s)}],\\
\omega_{31}(t,s) =& \frac{1}{\beta}[(v_{22}-v_{23})\lambda_1^{(t-s)} - (v_{21}-v_{23})\lambda_2^{(t-s)} + (v_{21}-v_{22})\lambda_3^{(t-s)}],\\
\omega_{12}(t,s) =& \frac{1}{\beta}[v_{11}(v_{13}-v_{12})\lambda_1^{(t-s)} - v_{12}(v_{12}-v_{13})\lambda_2^{(t-s)} + v_{13}(v_{12}-v_{11})\lambda_3^{(t-s)}],\\
\omega_{22}(t,s) =& \frac{1}{\beta}[v_{21}(v_{13}-v_{12})\lambda_1^{(t-s)} - v_{22}(v_{12}-v_{13})\lambda_2^{(t-s)} + v_{23}(v_{12}-v_{11})\lambda_3^{(t-s)}],\\
\omega_{32}(t,s) =& \frac{1}{\beta}[(v_{13}-v_{12})\lambda_1^{(t-s)} - (v_{12}-v_{13})\lambda_2^{(t-s)} + (v_{12}-v_{11})\lambda_3^{(t-s)}],\\
\omega_{13}(t,s) =& \frac{1}{\beta}[v_{11}(v_{12}v_{23}-v_{22}v_{13})\lambda_1^{(t-s)} - v_{12}(v_{11}v_{23}-v_{21}v_{13})\lambda_2^{(t-s)}\\
&+ v_{13}(v_{11}v_{22}-v_{21}v_{12})\lambda_3^{(t-s)}],\\
\omega_{23}(t,s) =& \frac{1}{\beta}[v_{21}(v_{12}v_{23}-v_{22}v_{13})\lambda_1^{(t-s)} - v_{22}(v_{11}v_{23}-v_{21}v_{13})\lambda_2^{(t-s)}\\
&+ v_{23}(v_{11}v_{22}-v_{21}v_{12})\lambda_3^{(t-s)}],\\
\omega_{33}(t,s) =& \frac{1}{\beta}[(v_{12}v_{23}-v_{22}v_{13})\lambda_1^{(t-s)} - (v_{11}v_{23}-v_{21}v_{13})\lambda_2^{(t-s)}\\
&+ (v_{11}v_{22}-v_{21}v_{12})\lambda_3^{(t-s)}]
\end{aligned} \tag{3.21}$$

and

$$\beta = (v_{11}v_{22} - v_{12}v_{21}) - (v_{11}v_{23} - v_{13}v_{21}) + (v_{12}v_{23} - v_{13}v_{22}).$$

From the formulae above, we obtain an expression for $e^{\underline{A}t}$ which is fixed and explicit once t is given.

Having explicit expressions for matrics $\underline{A}$ and $e^{\underline{A}t}$, we are ready to solve for matrix $\underline{\sigma}$ as the third step. The covariance equivalent principle is used in the multivariate model as well. From Equations (2.31) and (2.32), we know that the conditional variance and autocovariance of X_t in a multidimensional O-U process is directly dependent on $\underline{\Sigma}_{OU}(\underline{\Sigma}_{OU} = \underline{\sigma} \cdot \underline{\sigma}^T)$ instead of $\underline{\sigma}$ itself. Given $\underline{\Sigma}_{OU}$, there are multiple solutions for $\underline{\sigma}$ if a solution exists.

Without loss of generality we arbitrarily let $\underline{\sigma}$ be a lower triangular 3×3 matrix

$$\underline{\sigma} = \begin{bmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}.$$

With a lower triangular matrix $\underline{\sigma}$, we can reduce our calculations for $e^{\underline{A}(t-s)} \cdot \underline{\sigma}$. Letting matrix $\Omega(t,s)$ denote the product result, we have

$$\underline{\Omega}(t,s) = \begin{bmatrix} \omega_{11}(t,s) & \omega_{12}(t,s) & \omega_{13}(t,s) \\ \omega_{21}(t,s) & \omega_{22}(t,s) & \omega_{23}(t,s) \\ \omega_{31}(t,s) & \omega_{32}(t,s) & \omega_{33}(t,s) \end{bmatrix} \cdot \begin{bmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}. \quad (3.22)$$

and its elements can be obtained as

$$\begin{aligned}
\underline{\Omega}_{11}(t,s) &= \sigma_{11} \cdot \omega_{11}(t,s) + \sigma_{21} \cdot \omega_{12}(t,s) + \sigma_{31} \cdot \omega_{13}(t,s), \\
\underline{\Omega}_{12}(t,s) &= \sigma_{22} \cdot \omega_{12}(t,s) + \sigma_{32} \cdot \omega_{13}(t,s), \\
\underline{\Omega}_{13}(t,s) &= \sigma_{33} \cdot \omega_{13}(t,s), \\
\underline{\Omega}_{21}(t,s) &= \sigma_{11} \cdot \omega_{21}(t,s) + \sigma_{21} \cdot \omega_{22}(t,s) + \sigma_{31} \cdot \omega_{23}(t,s), \\
\underline{\Omega}_{22}(t,s) &= \sigma_{22} \cdot \omega_{22}(t,s) + \sigma_{32} \cdot \omega_{23}(t,s), \\
\underline{\Omega}_{23}(t,s) &= \sigma_{33} \cdot \omega_{23}(t,s), \\
\underline{\Omega}_{31}(t,s) &= \sigma_{11} \cdot \omega_{31}(t,s) + \sigma_{21} \cdot \omega_{32}(t,s) + \sigma_{31} \cdot \omega_{33}(t,s), \\
\underline{\Omega}_{32}(t,s) &= \sigma_{22} \cdot \omega_{32}(t,s) + \sigma_{32} \cdot \omega_{33}(t,s), \\
\underline{\Omega}_{33}(t,s) &= \sigma_{33} \cdot \omega_{33}(t,s). \qquad (3.23)
\end{aligned}$$

Therefore, we can write

$$\int_0^t e^{\underline{A}(t-s)} \cdot \underline{\sigma} d\underline{W}_s = \begin{bmatrix} \int_0^t \underline{\Omega}_{11}(t,s)dW_{1,s} + \int_0^t \underline{\Omega}_{12}(t,s)dW_{2,s} + \int_0^t \underline{\Omega}_{13}(t,s)dW_{3,s} \\ \int_0^t \underline{\Omega}_{21}(t,s)dW_{1,s} + \int_0^t \underline{\Omega}_{22}(t,s)dW_{2,s} + \int_0^t \underline{\Omega}_{23}(t,s)dW_{3,s} \\ \int_0^t \underline{\Omega}_{31}(t,s)dW_{1,s} + \int_0^t \underline{\Omega}_{32}(t,s)dW_{2,s} + \int_0^t \underline{\Omega}_{33}(1,s)dW_{3,s} \end{bmatrix}. \quad (3.24)$$

By the variance equivalent principle $Var(\underline{X}_1)$ equals $Cov(\int_0^1 e^{\underline{A}(1-s)} \underline{\sigma} d\underline{W}_s, \int_0^1 e^{\underline{A}(1-s)} \underline{\sigma} d\underline{W}_s)$ in a multi-dimensional O-U process, we can further expand the expression for each element of $Var(\underline{X}_1)$ with Ito formula

$$\begin{aligned}
Var(\underline{X}_1|\underline{X}_0) &= Cov\left[\int_0^1 \underline{\Omega}(1,s)d\underline{W}_s, \int_0^1 \underline{\Omega}(1,s)d\underline{W}_s\right], \\
[Var(\underline{X}_1|\underline{X}_0)]_{ij} &= \int_0^1 [\underline{\Omega}_{i1}(1,s)\underline{\Omega}_{j1}(1,s) + \underline{\Omega}_{i2}(1,s)\underline{\Omega}_{j2}(1,s) + \underline{\Omega}_{i3}(1,s)\underline{\Omega}_{j3}(1,s)]ds \quad (3.25)
\end{aligned}$$

where $1 \leq i \leq j \leq 3$. Given the symmetric property of the covariance matrix, we have $[Var(\underline{X}_1|\underline{X}_0)]_{21} = [Var(\underline{X}_1|\underline{X}_0)]_{12}$, $[Var(\underline{X}_1|\underline{X}_0)]_{31} = [Var(\underline{X}_1|\underline{X}_0)]_{13}$, $[Var(\underline{X}_1|\underline{X}_0)]_{32} = [Var(\underline{X}_1|\underline{X}_0)]_{23}$. From Equations (3.23) and (3.25), we know that all the integrations in the elements of $Var(\underline{X}_1)$ for this O-U process can be solved numerically to be a linear combination of $\sigma_{ij} \cdot \sigma_{i'j'}, (i, j, i', j' = 1, 2, 3)$. From Equations (2.26) and (2.38), $Var(\underline{X}_1)$ of the three-dimensional AR(1) process equals $\underline{\Sigma}_a$ whose elements have known numerical values. Using the equivalence between each elements of $Var(\underline{X}_1)$ of the AR(1) and O-U processes, six equations are established with six unknown variables $\sigma_{11}, \sigma_{21}, \sigma_{22}, \sigma_{31}, \sigma_{32}, \sigma_{33}$. In most cases, a numerical solution may be obtained by solving these equations. Therefore, a lower triangular matrix $\underline{\sigma}$ can be obtained.

With Equation (3.24), the expression of $Cov(\underline{X}_t, \underline{X}_s)$ $(t > s)$ can be expanded as

$$Cov(\underline{X}_t, \underline{X}_s|\underline{X}_0) = Cov(\int_0^t \underline{\Omega}(t,u)d\underline{W}_u, \int_0^s \underline{\Omega}(s,r)d\underline{W}_r),$$
$$[Cov(\underline{X}_t, \underline{X}_s)|\underline{X}_0]_{ij} = \int_0^s [\underline{\Omega}_{i1}(t,r)\underline{\Omega}_{j1}(s,r) + \underline{\Omega}_{i2}(t,r)\underline{\Omega}_{j2}(s,r) + \underline{\Omega}_{i3}(t,r)\underline{\Omega}_{j3}(s,r)]dr \tag{3.26}$$

where i, j = 1, 2, 3. Since we have solved the Σ_{OU} $(\sigma \cdot \sigma^T)$, a fixed numerical value is assigned to each element of the covariance matrix by solving the integrations in Equation (3.26).

3.3.2 Model Calculation

After estimating the model parameters, numerical values for $\underline{A}, \underline{\sigma}$ and an explicit expression for $e^{\underline{A}t}$ over t periods were determined in Section (3.3.1). From that, we can study the return rate of the whole portfolio. Assuming the weights of assets in the portfolio are p_1, p_2, p_3, the instantaneous rate of return of the portfolio, $X_{P,t}$ is

$$X_{P,t} = p_1 \cdot X_{1,t} + p_2 \cdot X_{2,t} + p_3 \cdot X_{3,t} \tag{3.27}$$

where $X_{i,t}$ $(i = 1, 2, 3)$ denotes the ith row of $\underline{X}_t$, which is a three-dimensional vector defined by the O-U process in our multivariate model. Furthermore, we have the conditional mean, variance and covariance functions of the instantaneous return rate for the portfolio at time t

$$E(X_{P,t}) = p_1 E(X_{1,t}) + p_2 E(X_{2,t}) + p_3 E(X_{3,t}), \tag{3.28}$$

$$\begin{aligned} Var(X_{P,t}) =& p_1^2 [Var(\underline{X}_t)]_{11} + p_2^2 [Var(\underline{X}_t)]_{22} + p_3^2 [Var(\underline{X}_t)]_{33} \\ &+ 2p_1p_2[Var(\underline{X}_t)]_{12} + 2p_1p_3[Var(\underline{X}_t)]_{13} + 2p_2p_3[Var(\underline{X}_t)]_{23} \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} Cov(X_{P,t}, X_{P,s}) = & \, p_1^2[Cov(\underline{X}_t, \underline{X}_s)]_{11} + p_2^2[Cov(\underline{X}_t, \underline{X}_s)]_{22} + p_3^2[Cov(\underline{X}_t, \underline{X}_s)]]_{33} \\ & + p_1p_2([Cov(\underline{X}_t, \underline{X}_s)]_{12} + [Cov(\underline{X}_t, \underline{X}_s)]_{21}) \\ & + p_1p_3([Cov(\underline{X}_t, \underline{X}_s)]_{13} + [Cov(\underline{X}_t, \underline{X}_s)]_{31}) \\ & + p_2p_3([Cov(\underline{X}_t, \underline{X}_s)]_{23} + [Cov(\underline{X}_t, \underline{X}_s)]_{32}) \end{aligned} \tag{3.30}$$

where $E(X_{i,t}|X_{i,0})$ $(i = 1, 2, 3)$ is the ith row of three-dimensional vector $[E(\underline{X}_t|\underline{X}_0)]$. $E(\underline{X}_t|\underline{X}_0)$ equals $e^{\underline{A}t}\underline{X}_0$ by Equation (2.30). $[Var(\underline{X}_t|\underline{X}_0)]_{ij}$ is the element in row i and column j of the 3×3 matrix $Var(\underline{X}_t|\underline{X}_0)$. And $[Cov(\underline{X}_t, \underline{X}_s|\underline{X}_0)]_{ij}$ is the element in row i and column j of the 3×3 matrix $Cov(\underline{X}_t, \underline{X}_s|\underline{X}_0)$. Here $Var(\underline{X}_t|\underline{X}_0)$ can be solved either by $\sum_{i=0}^{t-1} \underline{\Phi}^i \underline{\Sigma}_a (\underline{\Phi}^i)^T$ in Equation (2.26) of the AR(1) process or by Equation (3.25) of the O-U process. In the same way, the conditional covariance between $\underline{X}_t$ and $\underline{X}_s$ can be solved using Equation (2.27) of in the AR(1) process or Equation (2.32) for the O-U process.

It appears more convenient to use the equations in the AR(1) process to compute the conditional expected value and variance of the instantaneous return rate for the entire portfolio. However, the position process which is more interesting in practice has to be calculated from the O-U process. This is primarily because we assume that the investment rate is accumulated in a continuous way. Given matrices $\underline{A}$ and $\underline{\sigma}$ solved earlier, we set $\underline{B} = \begin{bmatrix} \underline{A} & \underline{0} \\ \underline{E} & \underline{0} \end{bmatrix}$ and $\underline{\sigma}_Y = \begin{bmatrix} \underline{\sigma} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix}$. Here $\underline{0}$ and $\underline{E}$ are 3×3 matrices, and hence both $\underline{B}$ and $\underline{\sigma}_Y$ are 6×6

matrices. Based on Equation (2.35), we have

$$\begin{bmatrix} X_{1,t}-\mu_1 \\ X_{2,t}-\mu_2 \\ X_{3,t}-\mu_3 \\ Y_{1,t}--\mu_1 t \\ Y_{2,t}--\mu_2 t \\ Y_{3,t}-\mu_3 t \end{bmatrix} = e^{\underline{B}t} \begin{bmatrix} X_{1,0}-\mu_1 \\ X_{2,0}-\mu_2 \\ X_{3,0}-\mu_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \int_0^t e^{\underline{B}(t-s)} \cdot \begin{bmatrix} \sigma_{11} & 0 & 0 & 0 & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 & 0 & 0 & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} d \begin{bmatrix} W_{1,s} \\ W_{2,s} \\ W_{3,s} \\ W_{4,s} \\ W_{5,s} \\ W_{6,s} \end{bmatrix}. \tag{3.31}$$

From the buildup of matrix $\underline{B}$, the eigenvalues of $\underline{B}$ are known as $0, 0, 0, \mu_1, \mu_2, \mu_3$, where μ_1, μ_2, μ_3 are the same eigenvalues as those for matrix $\underline{A}$. Hence, the eigenvalues of $e^{\underline{B}}$ are 1, 1, 1, λ_1, λ_2, and λ_3 ($\lambda_1{=}e^{\mu_1}$,$\lambda_2{=}e^{\mu_2}$,$\lambda_3{=}e^{\mu_3}$). The corresponding eigenvectors of matrix $e^{\underline{B}}$ are $\begin{bmatrix} 0 & 0 & 0 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ 0 & 0 & 0 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ 0 & 0 & 0 & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ 1 & 0 & 0 & \gamma_{41} & \gamma_{42} & \gamma_{43} \\ 0 & 1 & 0 & \gamma_{51} & \gamma_{52} & \gamma_{53} \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$.

In fact, $\begin{bmatrix} \gamma_{41} & \gamma_{42} & \gamma_{43} \\ \gamma_{51} & \gamma_{52} & \gamma_{53} \\ 1 & 1 & 1 \end{bmatrix}$ is also the eigenvector matrix of $e^{\underline{A}}$. By using the method for solving $e^{\underline{A}t}$ described in Section (3.3.1) , $e^{\underline{B}t}$ can be solved with more patience. For convenience, the following notation is used for the determinants

$$D_1 = \begin{vmatrix} \gamma_{22} & \gamma_{23} \\ \gamma_{32} & \gamma_{33} \end{vmatrix} = \gamma_{22}\gamma_{33} - \gamma_{23}\gamma_{32}, \qquad D_2 = \begin{vmatrix} \gamma_{21} & \gamma_{23} \\ \gamma_{31} & \gamma_{33} \end{vmatrix} = \gamma_{21}\gamma_{33} - \gamma_{23}\gamma_{31},$$

$$D_3 = \begin{vmatrix} \gamma_{22} & \gamma_{21} \\ \gamma_{32} & \gamma_{31} \end{vmatrix} = \gamma_{22}\gamma_{31} - \gamma_{21}\gamma_{32}, \qquad D_4 = \begin{vmatrix} \gamma_{12} & \gamma_{13} \\ \gamma_{32} & \gamma_{33} \end{vmatrix} = \gamma_{12}\gamma_{33} - \gamma_{13}\gamma_{32},$$

$$D_5 = \begin{vmatrix} \gamma_{11} & \gamma_{13} \\ \gamma_{31} & \gamma_{33} \end{vmatrix} = \gamma_{11}\gamma_{33} - \gamma_{13}\gamma_{31}, \qquad D_6 = \begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{31} & \gamma_{32} \end{vmatrix} = \gamma_{11}\gamma_{32} - \gamma_{12}\gamma_{31},$$

$$D_7 = \begin{vmatrix} \gamma_{13} & \gamma_{12} \\ \gamma_{23} & \gamma_{22} \end{vmatrix} = \gamma_{13}\gamma_{22} - \gamma_{12}\gamma_{23}, \qquad D_8 = \begin{vmatrix} \gamma_{11} & \gamma_{13} \\ \gamma_{21} & \gamma_{23} \end{vmatrix} = \gamma_{11}\gamma_{23} - \gamma_{13}\gamma_{21},$$

$$D_9 = \begin{vmatrix} \gamma_{11} & \gamma_{12} \\ \gamma_{21} & \gamma_{22} \end{vmatrix} = \gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}$$

and

$$D_{10} = \begin{vmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{vmatrix} = \gamma_{11}\gamma_{22}\gamma_{33} - \gamma_{11}\gamma_{23}\gamma_{32} + \gamma_{21}\gamma_{32}\gamma_{13} - \gamma_{21}\gamma_{12}\gamma_{33} + \gamma_{31}\gamma_{12}\gamma_{23} - \gamma_{31}\gamma_{22}\gamma_{13}.$$

Then $e^{\underline{B}t}$ can be expressed as

$$e^{\underline{B}t} = \begin{bmatrix} \omega_{11}(t,0) & \omega_{12}(t,0) & \omega_{13}(t,0) & 0 & 0 & 0 \\ \omega_{21}(t,0) & \omega_{22}(t,0) & \omega_{23}(t,0) & 0 & 0 & 0 \\ \omega_{31}(t,0) & \omega_{32}(t,0) & \omega_{33}(t,0) & 0 & 0 & 0 \\ \theta_{41}(t,0) & \theta_{42}(t,0) & \theta_{43}(t,0) & 1 & 0 & 0 \\ \theta_{51}(t,0) & \theta_{52}(t,0) & \theta_{53}(t,0) & 0 & 1 & 0 \\ \theta_{61}(t,0) & \theta_{62}(t,0) & \theta_{63}(t,0) & 0 & 0 & 1 \end{bmatrix} \tag{3.32}$$

where $\underline{\omega}(t,s)$ are the same series of functions used in Equation (3.20) to express $e^{\underline{A}t}$, and $\underline{\theta}(t,s)$ are the series of functions described below

$$
\begin{aligned}
\theta_{41}(t,s) &= \frac{1}{D_{10}}\left[D_1\gamma_{41}(\lambda_1^{(t-s)}-1) - D_2\gamma_{42}(\lambda_2^{(t-s)}-1) - D_3\gamma_{43}(\lambda_3^{(t-s)}-1)\right],\\
\theta_{51}(t,s) &= \frac{1}{D_{10}}\left[D_1\gamma_{51}(\lambda_1^{(t-s)}-1) - D_2\gamma_{52}(\lambda_2^{(t-s)}-1) - D_3\gamma_{53}(\lambda_3^{(t-s)}-1)\right],\\
\theta_{61}(t,s) &= \frac{1}{D_{10}}\left[D_1(\lambda_1^{(t-s)}-1) - D_2(\lambda_2^{(t-s)}-1) - D_3(\lambda_3^{(t-s)}-1)\right],\\
\theta_{42}(t,s) &= \frac{1}{D_{10}}\left[-D_4\gamma_{41}(\lambda_1^{(t-s)}-1) + D_5\gamma_{42}(\lambda_2^{(t-s)}-1) - D_6\gamma_{43}(\lambda_3^{(t-s)}-1)\right],\\
\theta_{52}(t,s) &= \frac{1}{D_{10}}\left[-D_4\gamma_{51}(\lambda_1^{(t-s)}-1) + D_5\gamma_{52}(\lambda_2^{(t-s)}-1) - D_6\gamma_{53}(\lambda_3^{(t-s)}-1)\right],\\
\theta_{62}(t,s) &= \frac{1}{D_{10}}\left[-D_4(\lambda_1^{(t-s)}-1) + D_5(\lambda_2^{(t-s)}-1) - D_6(\lambda_3^{(t-s)}-1)\right],\\
\theta_{43}(t,s) &= \frac{1}{D_{10}}\left[-D_7\gamma_{41}(\lambda_1^{(t-s)}-1) - D_8\gamma_{42}(\lambda_2^{(t-s)}-1) + D_9\gamma_{43}(\lambda_3^{(t-s)}-1)\right],\\
\theta_{53}(t,s) &= \frac{1}{D_{10}}\left[-D_7\gamma_{51}(\lambda_1^{(t-s)}-1) - D_8\gamma_{52}(\lambda_2^{(t-s)}-1) + D_9\gamma_{53}(\lambda_3^{(t-s)}-1)\right],\\
\theta_{63}(t,s) &= \frac{1}{D_{10}}\left[-D_7(\lambda_1^{(t-s)}-1) - D_8(\lambda_2^{(t-s)}-1) + D_9(\lambda_3^{(t-s)}-1)\right].
\end{aligned}
$$

Having an expression for $e^{\underline{B}(t-s)}$, $\underline{\Theta}(t,s) = e^{\underline{B}(t-s)}\underline{\sigma}_Y$ can be written as

$$
\underline{\Theta}(t,s) = \begin{bmatrix}
\omega_{11}(t,s) & \omega_{12}(t,s) & \omega_{13}(t,s) & 0 & 0 & 0\\
\omega_{21}(t,s) & \omega_{22}(t,s) & \omega_{23}(t,s) & 0 & 0 & 0\\
\omega_{31}(t,s) & \omega_{32}(t,s) & \omega_{33}(t,s) & 0 & 0 & 0\\
\theta_{41}(t,s) & \theta_{42}(t,s) & \theta_{43}(t,s) & 1 & 0 & 0\\
\theta_{51}(t,s) & \theta_{52}(t,s) & \theta_{53}(t,s) & 0 & 1 & 0\\
\theta_{61}(t,s) & \theta_{62}(t,s) & \theta_{63}(t,s) & 0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
\sigma_{11} & 0 & 0 & 0 & 0 & 0\\
\sigma_{21} & \sigma_{22} & 0 & 0 & 0 & 0\\
\sigma_{31} & \sigma_{32} & \sigma_{33} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{3.33}
$$

Therefore,

$$
\begin{aligned}
\underline{\Theta}_{11}(t,s) &= \underline{\Omega}_{11}(t,s) = \sigma_{11}\cdot\omega_{11}(t,s)+\sigma_{21}\cdot\omega_{12}(t,s)+\sigma_{31}\cdot\omega_{13}(t,s),\\
\underline{\Theta}_{12}(t,s) &= \underline{\Omega}_{12}(t,s) = \sigma_{22}\cdot\omega_{12}(t,s)+\sigma_{32}\cdot\omega_{13}(t,s),\\
\underline{\Theta}_{13}(t,s) &= \underline{\Omega}_{13}(t,s) = \sigma_{33}\cdot\omega_{13}(t,s),\\
\underline{\Theta}_{21}(t,s) &= \underline{\Omega}_{21}(t,s) = \sigma_{11}\cdot\omega_{21}(t,s)+\sigma_{21}\cdot\omega_{22}(t,s)+\sigma_{31}\cdot\omega_{23}(t,s),\\
\underline{\Theta}_{22}(t,s) &= \underline{\Omega}_{22}(t,s) = \sigma_{22}\cdot\omega_{22}(t,s)+\sigma_{32}\cdot\omega_{23}(t,s),\\
\underline{\Theta}_{23}(t,s) &= \underline{\Omega}_{23}(t,s) = \sigma_{33}\cdot\omega_{23}(t,s),\\
\underline{\Theta}_{31}(t,s) &= \underline{\Omega}_{31}(t,s) = \sigma_{11}\cdot\omega_{31}(t,s)+\sigma_{21}\cdot\omega_{32}(t,s)+\sigma_{31}\cdot\omega_{33}(t,s),\\
\underline{\Theta}_{32}(t,s) &= \underline{\Omega}_{32}(t,s) = \sigma_{22}\cdot\omega_{32}(t,s)+\sigma_{32}\cdot\omega_{33}(t,s),\\
\underline{\Theta}_{33}(t,s) &= \underline{\Omega}_{33}(t,s) = \sigma_{33}\cdot\omega_{33}(t,s),\\
\underline{\Theta}_{41}(t,s) &= \sigma_{11}\cdot\theta_{41}(t,s)+\sigma_{21}\cdot\theta_{42}(t,s)+\sigma_{31}\cdot\theta_{43}(t,s),\\
\underline{\Theta}_{42}(t,s) &= \sigma_{22}\cdot\theta_{42}(t,s)+\sigma_{32}\cdot\theta_{43}(t,s),\\
\underline{\Theta}_{43}(t,s) &= \sigma_{33}\cdot\theta_{43}(t,s),\\
\underline{\Theta}_{51}(t,s) &= \sigma_{11}\cdot\theta_{51}(t,s)+\sigma_{21}\cdot\theta_{52}(t,s)+\sigma_{31}\cdot\theta_{53}(t,s),\\
\underline{\Theta}_{52}(t,s) &= \sigma_{22}\cdot\theta_{52}(t,s)+\sigma_{32}\cdot\theta_{53}(t,s),\\
\underline{\Theta}_{53}(t,s) &= \sigma_{33}\cdot\theta_{53}(t,s),\\
\underline{\Theta}_{61}(t,s) &= \sigma_{11}\cdot\theta_{61}(t,s)+\sigma_{21}\cdot\theta_{62}(t,s)+\sigma_{31}\cdot\theta_{63}(t,s),\\
\underline{\Theta}_{62}(t,s) &= \sigma_{22}\cdot\theta_{62}(t,s)+\sigma_{32}\cdot\theta_{63}(t,s),\\
\underline{\Theta}_{63}(t,s) &= \sigma_{33}\cdot\theta_{63}(t,s).
\end{aligned} \tag{3.34}
$$

The elements in the last three columns of $\underline{\Theta}(t,s)$ are all 0. The series of $\underline{\Omega}(t,s)$ functions in the above equations are the same as those used to express $e^{\underline{A}(t-s)}$.
Therefore, we can write

$$
\int_0^t e^{\underline{B}(t-s)}\cdot\underline{\sigma}_Y d\underline{W}_s = \begin{bmatrix}
\int_0^t \underline{\Omega}_{11}(t,s)dW_{1,s}+\int_0^t \underline{\Omega}_{12}(t,s)dW_{2,s}+\int_0^t \underline{\Omega}_{13}(t,s)dW_{3,s}\\
\int_0^t \underline{\Omega}_{21}(t,s)dW_{1,s}+\int_0^t \underline{\Omega}_{22}(t,s)dW_{2,s}+\int_0^t \underline{\Omega}_{23}(t,s)dW_{3,s}\\
\int_0^t \underline{\Omega}_{31}(t,s)dW_{1,s}+\int_0^t \underline{\Omega}_{32}(t,s)dW_{2,s}+\int_0^t \underline{\Omega}_{33}(1,s)dW_{3,s}\\
\int_0^t \underline{\Theta}_{41}(t,s)dW_{1,s}+\int_0^t \underline{\Theta}_{42}(t,s)dW_{2,s}+\int_0^t \underline{\Theta}_{43}(t,s)dW_{3,s}\\
\int_0^t \underline{\Theta}_{51}(t,s)dW_{1,s}+\int_0^t \underline{\Theta}_{52}(t,s)dW_{2,s}+\int_0^t \underline{\Theta}_{53}(t,s)dW_{3,s}\\
\int_0^t \underline{\Theta}_{61}(t,s)dW_{1,s}+\int_0^t \underline{\Theta}_{62}(t,s)dW_{2,s}+\int_0^t \underline{\Theta}_{63}(1,s)dW_{3,s}
\end{bmatrix} \tag{3.35}
$$

In Equation (3.35), the first three rows illustrate the stochastic variate $\underline{X}_t$, which is exactly the same as that obtained in Equation (3.24) with $\underline{X}_t$ only. The last three rows illustrate the stochastic variate $\underline{Y}_t$. Because the uncertainty in $\underline{Y}_t$ comes entirely from the randomness of $\underline{X}_t$, three Brownian Motions are sufficient to characterize the random movements of both $\underline{X}_t$ and $\underline{Y}_t$. Since $\begin{bmatrix} \underline{X}_t \\ \underline{Y}_t \end{bmatrix} = e^{\underline{B}t} \begin{bmatrix} \underline{X}_0 \\ \underline{0} \end{bmatrix} + \int_0^t e^{\underline{B}(t-s)} \cdot \underline{\sigma}_Y d\underline{W}s$, the mean of $\underline{Y}_t$ can be calculated as

$$E \begin{bmatrix} Y_{1,t}|\underline{X}_0 \\ Y_{2,t}|\underline{X}_0 \\ Y_{3,t}|\underline{X}_0 \end{bmatrix} = \begin{bmatrix} \theta_{41}(t,0)X_{1,0} + \theta_{42}(t,0)X_{2,0} + \theta_{43}(t,0)X_{3,0} \\ \theta_{51}(t,0)X_{1,0} + \theta_{52}(t,0)X_{2,0} + \theta_{53}(t,0)X_{3,0} \\ \theta_{61}(t,0)X_{1,0} + \theta_{62}(t,0)X_{2,0} + \theta_{63}(t,0)X_{3,0} \end{bmatrix}. \tag{3.36}$$

With Equation (3.35) and Ito formula, the matrix of $Cov(\underline{Y}_t, \underline{Y}_s)$ $(t > s)$ can be expanded as

$$\begin{aligned}
Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0) &= Cov\left[\int_0^t \underline{\Theta}(t,u)d\underline{W}_u, \int_0^s \underline{\Theta}(s,r)d\underline{W}_r\right], \\
[Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{11} &= \int_0^s [\underline{\Theta}_{41}(t,r)\underline{\Theta}_{41}(s,r) + \underline{\Theta}_{42}(t,r)\underline{\Theta}_{42}(s,r) + \underline{\Theta}_{43}(t,r)\underline{\Theta}_{43}(s,r)]dr, \\
[Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{12} &= \int_0^s [\underline{\Theta}_{41}(t,r)\underline{\Theta}_{51}(s,r) + \underline{\Theta}_{42}(t,r)\underline{\Theta}_{52}(s,r) + \underline{\Theta}_{43}(t,r)\underline{\Theta}_{53}(s,r)]dr, \\
[Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{13} &= \int_0^s [\underline{\Theta}_{41}(t,r)\underline{\Theta}_{61}(s,r) + \underline{\Theta}_{42}(t,r)\underline{\Theta}_{62}(s,r) + \underline{\Theta}_{43}(t,r)\underline{\Theta}_{63}(s,r)]dr, \\
[Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{21} &= \int_0^s [\underline{\Theta}_{51}(t,r)\underline{\Theta}_{41}(s,r) + \underline{\Theta}_{52}(t,r)\underline{\Theta}_{42}(s,r) + \underline{\Theta}_{53}(t,r)\underline{\Theta}_{43}(s,r)]dr, \\
[Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{22} &= \int_0^s [\underline{\Theta}_{51}(t,r)\underline{\Theta}_{51}(s,r) + \underline{\Theta}_{52}(t,r)\underline{\Theta}_{52}(s,r) + \underline{\Theta}_{53}(t,r)\underline{\Theta}_{53}(s,r)]dr, \\
[Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{23} &= \int_0^s [\underline{\Theta}_{51}(t,r)\underline{\Theta}_{61}(s,r) + \underline{\Theta}_{52}(t,r)\underline{\Theta}_{62}(s,r) + \underline{\Theta}_{53}(t,r)\underline{\Theta}_{63}(s,r)]dr, \\
[Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{31} &= \int_0^s [\underline{\Theta}_{61}(t,r)\underline{\Theta}_{41}(s,r) + \underline{\Theta}_{62}(t,r)\underline{\Theta}_{42}(s,r) + \underline{\Theta}_{63}(t,r)\underline{\Theta}_{43}(s,r)]dr, \\
[Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{32} &= \int_0^s [\underline{\Theta}_{61}(t,r)\underline{\Theta}_{51}(s,r) + \underline{\Theta}_{62}(t,r)\underline{\Theta}_{52}(s,r) + \underline{\Theta}_{63}(t,r)\underline{\Theta}_{53}(s,r)]dr, \\
[Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{33} &= \int_0^s [\underline{\Theta}_{61}(t,r)\underline{\Theta}_{61}(s,r) + \underline{\Theta}_{62}(t,r)\underline{\Theta}_{62}(s,r) + \underline{\Omega}_{63}(t,r)\underline{\Omega}_{63}(s,r)]dr.
\end{aligned} \tag{3.37}$$

This expression for the covariance matrix of $\underline{Y}_t$ looks cumbersome. However, all the parameters ($\underline{\sigma}$, the eigenvalues and eigenvectors of $\underline{B}$) in the expression have been determined to be fixed numbers for any particular estimated model. The integrations shown above can be solved numerically in a short time once t and s are given. Therefore, we have provided a way of calculating the conditional variance and covariance of the accumulated function of the return rate in a short time scale which avoids using simulations.

Finally, expressions for the conditional expected value, variance and covariance functions of $Y_{P,t}$, the accumulated function of the rate of the return for the whole asset portfolio, can be derived as

$$\begin{aligned} E(Y_{P,t}|\underline{X}_0) =& p_1 E(Y_{1,t}|\underline{X}_0) + p_2 E(Y_{2,t}|\underline{X}_0) + p_3 E(Y_{3,t}|\underline{X}_0), \quad (3.38)\\ Var(Y_{P,t}|\underline{X}_0) =& p_1^2[Var(\underline{Y}_t|\underline{X}_0)]_{11} + p_2^2[Var(\underline{Y}_t|\underline{X}_0)]_{22} + p_3^2[Var(\underline{Y}_t|\underline{X}_0)]_{33} \\ &+ 2p_1p_2[Var(\underline{Y}_t|\underline{X}_0)]_{12} + 2p_1p_3[Var(\underline{Y}_t|\underline{X}_0)]_{13} + 2p_2p_3[Var(\underline{Y}_t|\underline{X}_0)]_{23} \quad (3.39) \end{aligned}$$

and

$$\begin{aligned} Cov(Y_{P,t}, Y_{P,s}|\underline{X}_0) =& p_1^2[Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{11} + p_2^2[Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{22} + p_3^2[Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]]_{33} \\ &+ p_1p_2([Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{12} + [Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{21}) \\ &+ p_1p_3([Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{13} + [Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{31}) \\ &+ p_2p_3([Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{23} + [Cov(\underline{Y}_t, \underline{Y}_s|\underline{X}_0)]_{32}) \quad (3.40) \end{aligned}$$

In Equations (3.39) and (3.40), the correlations among the assets are fully considered.

In this chapter, we havepresented an approach that can be used to analyze a three-asset portfolio by symbolic and numerical calculations. Theoretically, this method can be extended to a portfolio containing more assets but with greater complexity and more computation time. Occasionally, round-off errors present a significant problem. In practice, symbolic calculations and simplification should be performed before plugging in any numerical values.

Chapter 4

Fitting and Comparing Investment Models

In this chapter, three asset portfolios, which are assumed to be composed of a long-term bond, a short-term Treasury bill and an equity, are used as examples for calculations and illustrations. Among them, Asset Portfolio 1 is assumed to be 60% long-term bond, 30% short-term Treasury bill and 10 % equity. Asset Portfolio 2 is assumed to be 60% long-term bond, 10% short-term Treasury bill and 30% equity. As an extreme case, Asset Portfolio 3 is assumed to be a high-risk package which is 30% long-term bond, 10% short-term Treasury bill and 60% equity. The parameters of the models are estimated from historical data for each asset. Moreover, the conditional means and variances of X_t and Y_t of the entire portfolio are calculated and compared under these three models.

4.1 Fitting the Models

The parameter estimates of a financial model can be greatly dependent on the time interval and the period over which historical data is collected. How to select proper data to estimate a model is a highly debatable topic in the insurance and financial fields. This project focuses on the analysis of different mathematical models rather than data selection methods. In this project, the past observations of the rates of return on ten-year Constant Maturity Treasury Bills, three-month Treasury Bills and the S&P 500 Index in the US market are

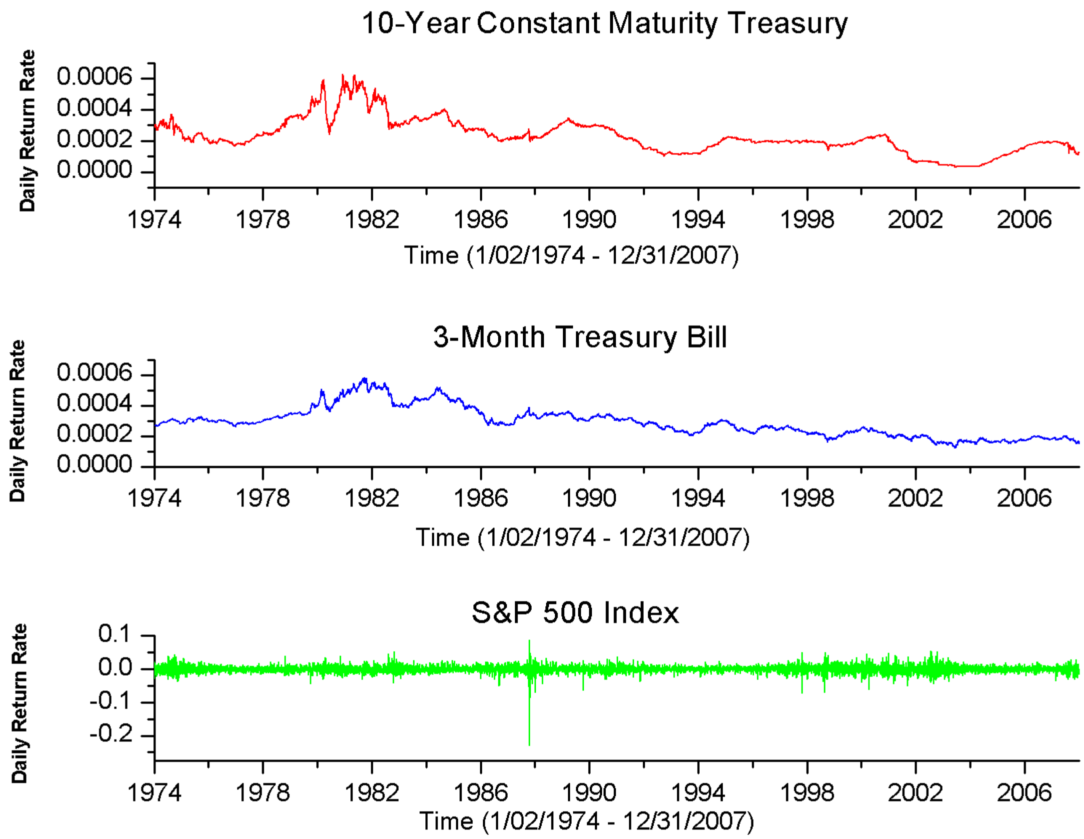

Figure 4.1: The Historical Return Rates of Three Assets

taken as proxies for the return rates of the long-term bond, the short-term Treasury bill and the equity in the portfolio, respectively. Daily return rates (shown in Figure 4.1) for these three financial instruments from 01/02/1974 to 12/31/2007 were obtained Federal Reserve Statistical Release and Yahoo Finance. All the univariate and multivariate processes in the three invest models are fitted to these data by R Package with ordinary least squares method. We then compare the features of these three models.

4.1.1 Testing the Models

The first step is to fit AR(1) processes to historical data. Therefore, we plot the autocorrelation functions (ACFs) and partial autocorrelation functions (PACFs) for the daily return rates of these three assets to determine whether an AR(1) process is appropriate (Figure 4.2). Additionally, the ACF and PACF of the daily return rate of Portfolio 1 are plotted to test the feasibility of applying the global model. In Figure 4.2, the ACFs of the data for the long-term bond and short-term Treasury bill yield a very slow decay, and the strong correlations exist even after a 40-day lag. The PACFs of these two time series quickly decay to 0 after lag 1. Therefore, the ACFs and PACFs suggest that AR(1) models should be used to fit the daily return rates of the long-term bond and short-term bill. Both the ACF and PACF of the daily return rate of the equity are close to 0 for lags more than one day. The weak correlations among the time series indicate that the daily return rate for the equity has a very weak daily relationship and almost approaches white noise. Surprisingly, the ACF and PACF of the daily return rate of Portfolio 1 are nearly 0 for lags more than one day, which means the process of the daily return rate of the portfolio should also be modeled as white noise. However, only 10% of the portfolio is equity which is close to white noise. The other 90% is invested in long-term bonds and short-term Treasury bills, which have strong daily relationships. From Figure 4.1, we can see that there is a much higher daily volatility for the equity than for the long-term bond and short-term bill. Statistic calculations show that the daily volatility of the equity is more than 100 times higher than that of the long-term bond and short-term bill. As a result, the noise of the 10% equity overwhelms all the relationship among the assets and makes the return rate of the whole portfolio behave like white noise. Due to the high volatility and weak correlation of the return rate of the equity, it is difficult to approximate the ϕ of the AR(1) processes for the daily return rate of the equity or the daily return rate of Portfolio 1 calculated in the global model. It is also difficult to characterize the relationship between the equity and the other assets. Therefore, an "Annual Method" is used to reduce the noise in the rate for the equity. In the Annual Method, instead of modeling the AR(1) processes directly from the daily data (which is called "Daily Method" in this project), the daily data in each year is summarized into be an annual rate and then all the past annual rates are used to model the AR(1) process. The results for these two methods are shown in Table 4.1. To make the results comparable, all the parameters obtained by the Daily Method are converted into annual unit. The ϕ

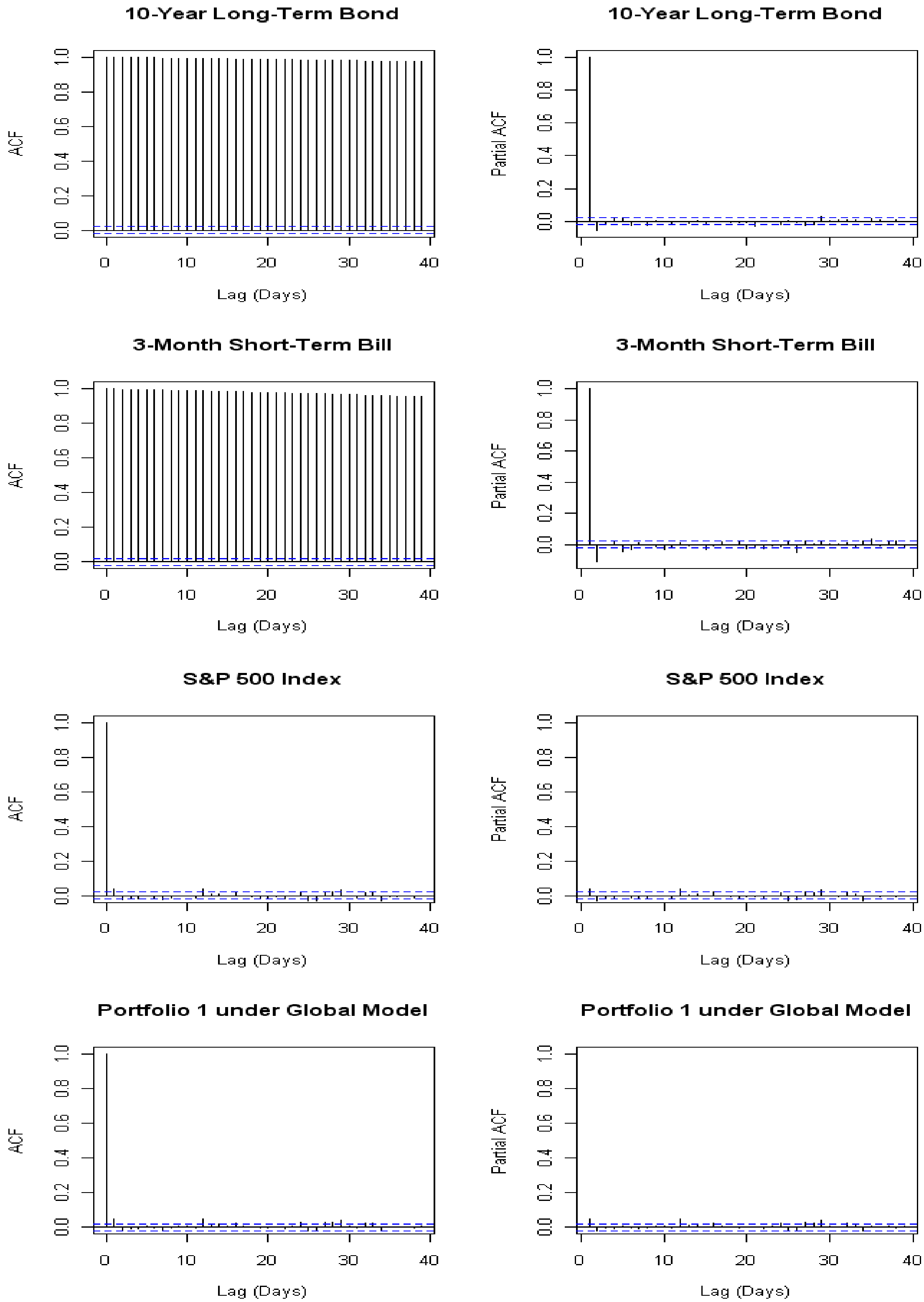

Figure 4.2: Autocorrelation Function and Partial Autocorrelation Function of the Daily Return Rates

and σ_a of the AR(1) processes for the rates of the long-term bond and short-term Treasury bill are very close for these two methods (0.929 vs 0.931 and 0.857 vs 0.852). Since there is a high correlation in the daily data of each of these two assets, the annual volatilities (σ) obtained from the Daily Method are only slightly higher than those obtained directly from the Annual Method. Most of the noise cannot be canceled by summing the daily rates for these two assets. However, by using the Annual Method, the ϕ of the AR(1) process modeling the annual rate of the equity increases from 0 to 0.06, and the ϕ of the AR(1) process modeling the annual rate of the portfolio under the global model greatly increases from 0 to 0.766. The annual volatilities for the rates of both the equity and the portfolio decrease more than tenfold in the Annual Method. Since the daily rates of the equity are almost independent, the summation of the daily data into annual data effectively reduces the total variation and make it easier to identify the correlations between the equity and other assets. The ACFs and PACFs of the rates collected by the Annual Method are

Table 4.1: Comparison of Daily Method and Annual Method

	Daily Method				Annual Method			
	Long[1]	Short[2]	Equity[3]	Portfolio[4]	Long	Short	Equity	Portfolio
ϕ	0.929	0.859	0	0	0.932	0.852	0.06	0.766
σ_a	0.011	0.015	2.535	0.255	0.010	.014	0.129	0.0175

[1] Long = the rate of the long-term bond
[2] Short = the rate of the short-term Treasury bill
[3] Equity = the rate of the equity
[4] Portfolio = the rate of the portfolio under the global model

calculated and shown in Figure 4.3. The time series of the annual rate of the equity remains close to white noise since its ACF and PACF is around 0 for all lags after lag 0. The ACFs of the rates of the long-term bond, the short-term Treasury bill and the portfolio under the global model yield a slow geometric decay, and the PACFs quickly decreases after one-year lag. Therefore, it is reasonable to fit AR(1) processes in our model.

4.1.2 Fitting the Univariate Model

In the univariate model, the three assets in the portfolio are modeled by three separate AR(1) processes. By fitting the data with the Annual Method, we have:

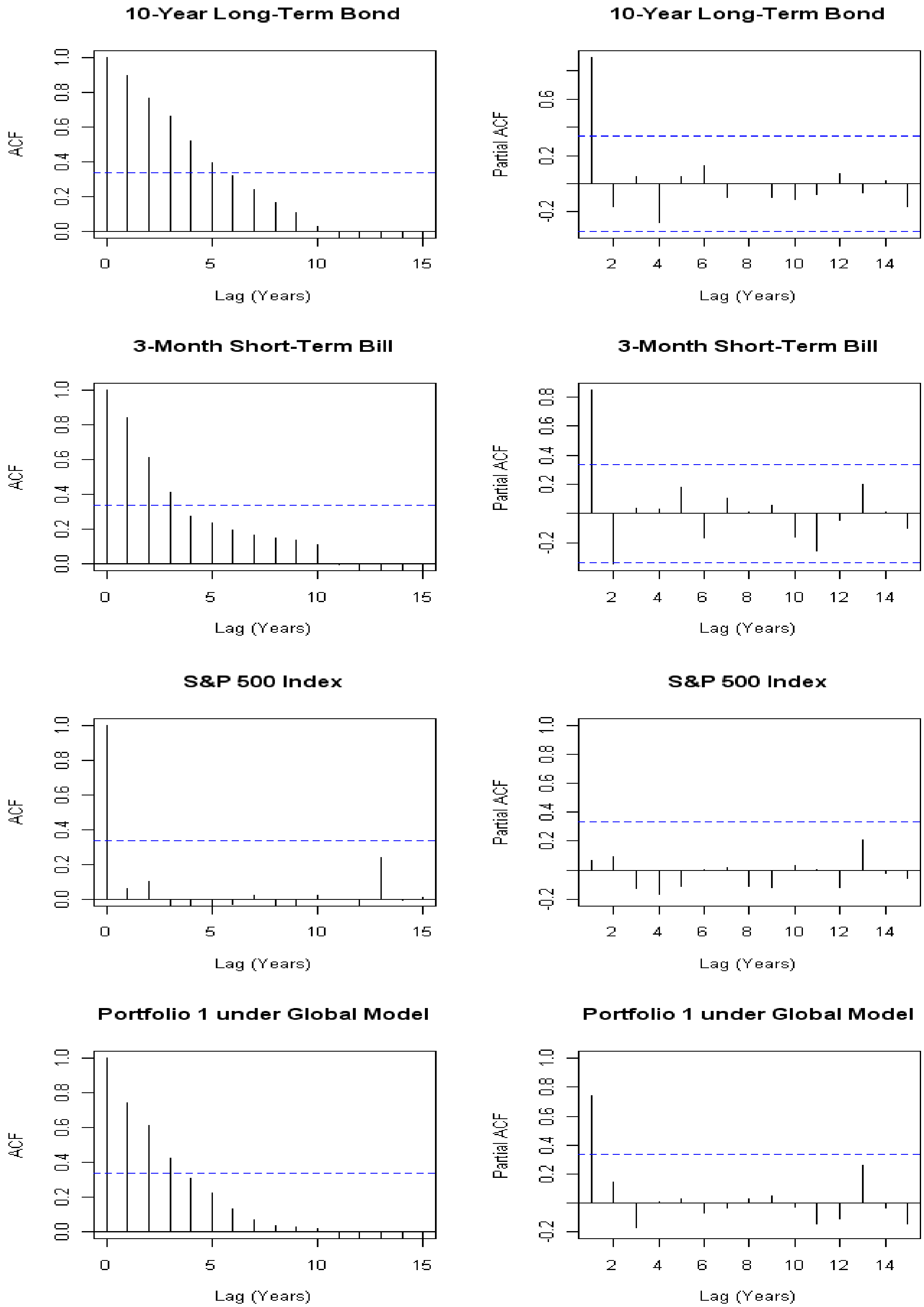

Figure 4.3: Autocorrelation Function and Partial Autocorrelation Function of the Annual Return Rates

- the AR(1) process for the ten-year long-term bond

$$X_{L,t} - 0.07263607 = 0.9316741(X_{L,t-1} - 0.07263607) + a_{L,t} \tag{4.1}$$

where 0.07263607 is the long-term mean of the annual return rate for the long-term bond, and the random error term $a_{L,t} \sim N(0, 9.398E - 05)$;

- the AR(1) process for the three-month short-term Treasury bill

$$X_{S,t} - 0.05735164 = 0.851751(X_{S,t-1} - 0.05735164) + a_{S,t} \tag{4.2}$$

where 0.05735164 is the long-term mean of the annual return rate for the short-term Treasury bill, and the random error term $a_{S,t} \sim N(0, 1.923E - 04)$; and,

- the AR(1) process for the equity

$$X_{E,t} - 0.08146797 = 0.06291437(X_{E,t-1} - 0.08146797) + a_{E,t} \tag{4.3}$$

where 0.08146797 is the long-term mean of the annual return rate for the equity, and the random error term $a_{E,t} \sim N(0, 0.01673)$.

Among these three assets, the short-term Treasury bill has the lowest average return rate (5.74%) in the long term, while the equity has the highest average return rate (8.15%) in long term. As described in Section 4.1.1, the annual rates of the long-term bond and short-term Treasury bill follow the AR(1) processes with their ϕ quite close to 1, which indicates a strong correlation between the rates of the current year and the previous year. The AR(1) process for the equity has a ϕ close to 0 and it is meaningless to estimate the return rate of the equity this year from the rate of the past year.

The ϕ for all the assets are between 0 and 1. Therefore, the AR(1) processes can be converted to their corresponding O-U processes. From Equations (2.21) and (2.22), we know that $\alpha = \frac{-\ln(\phi)}{\Delta}$ and $\sigma = \sqrt{\frac{2\alpha\sigma_a^2}{1-\phi^2}}$. Hence, we have the following covariance equivalent O-U processes:

- the O-U process for the ten-year long-term bond

$$X_{L,t} - 0.07263607 = e^{-\alpha_L t}(X_{L,0} - 0.07263607) + \sigma_L \int_0^t e^{-\alpha_L(t-s)} dW_s \tag{4.4}$$

where α_L and σ_L in this equation equal 0.07077217 and 0.01003915, respectively.

- the O-U process for the three-month short-term Treasury bill

$$X_{S,t} - 0.05735164 = e^{-\alpha_S t}(X_{S,t-1} - 0.05735164) + \sigma_S \int_0^t e^{-\alpha_S (t-s)} dW_s \quad (4.5)$$

where α_S and σ_S in this equation equal 0.1604602 and 0.01499311, respectively.

- the O-U process for the equity

$$X_{E,t} - 0.08146797 = e^{-\alpha_E t}(X_{E,t-1} - 0.08146797) + \sigma_E \int_0^t e^{-\alpha_E (t-s)} dW_s \quad (4.6)$$

where α_E and σ_E in this equation equal 2.765981 and 0.3047845, respectively.

Note that α is the market force bringing the rate of return to its long-term mean. It will take $1/\alpha$ units of time to reduce the distance that the rate is away from its long-term mean by 63.2% (on average). As a result, it takes about 42, 19 and 1 year for the conational expected values of long-term bond, short-term Treasury bill, and equity respectively to get 95 % closer to their long-term equilibrium (i.e. reducing the distance away from the long-term mean by 95 %).

4.1.3 Fitting the Multivariate Model

The three assets can be also modeled by a multidimensional AR(1) process in the multivariate model, given by

$$\begin{bmatrix} X_{L,t} - 0.0726 \\ X_{S,t} - 0.0574 \\ X_{E,t} - 0.0815 \end{bmatrix} = \underline{\Phi} \cdot \begin{bmatrix} X_{L,t-1} - 0.0726 \\ X_{S,t-1} - 0.0574 \\ X_{E,t-1} - 0.0815 \end{bmatrix} + \begin{bmatrix} \hat{a}_{L,t} \\ \hat{a}_{S,t} \\ \hat{a}_{E,t} \end{bmatrix} \quad (4.7)$$

where $\underline{\Phi} = \begin{bmatrix} 0.56671187 & 0.3501652 & -0.004155868 \\ -0.04221515 & 0.8784722 & 0.014827817 \\ 2.20449749 & -0.8540416 & 0.040385848 \end{bmatrix}$, and $[\hat{a}_{L,t}, \hat{a}_{S,t}, \hat{a}_{L,t}]^T$ follows a multivariate normal distribution with mean $\underline{0}$ and 3×3 covariance matrix

$$\underline{\Sigma}_a = \begin{bmatrix} 7.950197e-05 & 9.764117e-05 & -0.0002809399 \\ 9.764117e-05 & 1.884875e-04 & -0.0001513304 \\ -2.809399e-04 & -1.513304e-04 & 0.0157239355 \end{bmatrix}.$$

To identify the relationships among the assets clearly, every two assets of these three assets are fitted separately by an two-dimensional AR(1) process. We obtain three two-dimensional AR(1) processes:

- the AR(1) process for the ten-year long-term bond and three-month Treasury bill

$$\begin{bmatrix} X_{L,t} - 0.0726 \\ X_{S,t} - 0.0574 \end{bmatrix} = \begin{bmatrix} 0.5650 & 0.3496 \\ -0.03609 & 0.8805 \end{bmatrix} \begin{bmatrix} X_{L,t-1} - 0.0726 \\ X_{S,t-1} - 0.0574 \end{bmatrix} + \begin{bmatrix} \hat{a}_{L,t} \\ \hat{a}_{S,t} \end{bmatrix} \quad (4.8)$$

  where $[\hat{a}_{L,t}, \hat{a}_{S,t}]^T$ follows a multivariate normal distribution with a covariance matrix $\begin{bmatrix} 7.979e-05 & 9.661e-05 \\ 9.661e-05 & 1.922e-4 \end{bmatrix}$;

- the AR(1) process for the ten-year long-term bond and equity

$$\begin{bmatrix} X_{L,t} - 0.0726 \\ X_{E,t} - 0.0815 \end{bmatrix} = \begin{bmatrix} 0.9338 & -0.003817 \\ 1.3092 & 0.039560 \end{bmatrix} \begin{bmatrix} X_{L,t-1} - 0.0726 \\ X_{E,t-1} - 0.0815 \end{bmatrix} + \begin{bmatrix} \hat{a}_{L,t}, \hat{a}_{E,t} \end{bmatrix} \quad (4.9)$$

  where $[\hat{a}_{L,t}, \hat{a}_{E,t}]^T$ has the covariance matrix $\begin{bmatrix} 9.373e-05 & -3.156e-4 \\ -3.156e-04 & 0.0158086 \end{bmatrix}$; and

- the AR(1) process for the three-month short-term Treasury bill and equity

$$\begin{bmatrix} X_{S,t} - 0.0574 \\ X_{E,t} - 0.0815 \end{bmatrix} = \begin{bmatrix} 0.8448 & 0.01474 \\ 0.9022 & 0.04517 \end{bmatrix} \begin{bmatrix} X_{S,t-1} - 0.0574 \\ X_{E,t-1} - 0.0815 \end{bmatrix} + \begin{bmatrix} \hat{a}_{S,t} \\ \hat{a}_{E,t} \end{bmatrix} \quad (4.10)$$

  where $[\hat{a}_{S,t}\hat{a}_{E,t}]^T$ has the covariance matrix $\begin{bmatrix} 1.886e-04 & -1.595e-4 \\ -1.595e-04 & 0.0161525 \end{bmatrix}$.

Based on the two-dimensional AR(1) process described in Equation (4.8), the return rate of the long-term bond depends on 56.5% of the return rate of the long-term bond last year and 35% of the return rate of short-term Treasury bill last year. On the other hand, the return rate of the short-term Treasury bill is mainly dependent on the rate of the short-term Treasury bill last year (88%), with almost no dependence on the rate of the long-term bond last year (only -3.6%). This is reasonable since the short-term interest rate is an important consideration in determining the long-term interest rate. However, the short-term interest rate is influenced by a number of other short term economic factors and is not directly affected by the long-term interest rate. The AR(1) processes described in Equations (4.9) and (4.10) show that the return rate of the equity is greatly dependent on the past long- or short-term interest rates but not on the past return rate of the equity itself. On the other hand, the past return rate of the equity has almost no impact on the long- and short-term interest rates. By comparing Equations (4.8), (4.9) and (4.10) which describe the two-dimensional

AR(1) processes to Equation (4.7) describing the three-dimensional AR(1) process, we find that the three-dimensional AR(1) process in Equation (4.7) precisely defines the same relationship between the return rates of the long-term bond and the short-term Treasury bill. However, it suggests different parameters for characterizing the relationships between equity and the other assets. Due to the correlation between the return rates of the long-term bond and the short-term Treasury bill, the dependence of the rate of the equity can be described as a series of different combinations of the past return rates of the long-term bond and the short-term Treasury bill. Therefore, it is difficult to determine the relationship of the equity with the long-term bond and short-term Treasury bill directly from the three-dimensional AR(1) process. However, all these different combinations generate similiar results for the return rate of the equity.

As described in Section 3.3.1, the multivariate AR(1) process can be converted into a covariance equivalent three-dimensional O-U process. Here, we have the equation

$$\begin{bmatrix} X_{L,t} - 0.0726 \\ X_{S,t} - 0.0574 \\ X_{E,t} - 0.0815 \end{bmatrix} = e^{\underline{A}t} \begin{bmatrix} X_{L,0} - 0.0726 \\ X_{S,0} - 0.0574 \\ X_{E,0} - 0.0815 \end{bmatrix} + \int_0^t e^{\underline{A}(t-s)} \underline{\sigma} d\underline{W}s \tag{4.11}$$

where $\underline{A} = \begin{bmatrix} -0.4664490 & 0.43537486 & -0.03252307 \\ -0.1564092 & -0.05791841 & 0.04284787 \\ 8.4957912 & -4.92206720 & -2.52848895 \end{bmatrix}$,

and $\underline{\sigma} =$
$$\begin{bmatrix} 0.007739484 & 0 & 0 \\ 0.011560909 & 0.009628814 & 0 \\ -0.059952797 & -0.031578778 & 0.2800872 \end{bmatrix}.$$

To obtain a closed-form expression for $e^{\underline{A}t}$ in Equation (3.20), the eigenvalues (μ_1, μ_2 and μ_3) and the corresponding eigenvectors of $\underline{A}$ are calculated. We have

$$\mu_1 = -0.1504782, \ \ \mu_2 = -0.6546244, \ \ \mu_3 = -2.2477537, \tag{4.12}$$

and

$$\begin{bmatrix} v_{11} & v_{21} & 1 \\ v_{12} & v_{22} & 1 \\ v_{13} & v_{23} & 1 \end{bmatrix} = \begin{bmatrix} 0.5576568 & 0.4794176 & 1 \\ 0.21100548 & -0.01649836 & 1 \\ 0.02264505 & -0.01794929 & 1 \end{bmatrix}. \tag{4.13}$$

μ_1, μ_2, and μ_3 are all negative values in Equation (4.12), so the three-dimensional O-U process is stationary.

To calculate the first and second moments of $\underline{Y}_t$, we need to construct the matrices $\underline{B}$ and $\underline{\sigma}_Y$ are constructed from matrices$\underline{A}$ and $\underline{\sigma}$. We have

$$\underline{B} = \begin{bmatrix} \underline{A} & \underline{0} \\ \underline{E} & \underline{0} \end{bmatrix} = \begin{bmatrix} -0.4664490 & 0.43537486 & -0.03252307 & 0 & 0 & 0 \\ -0.1564092 & -0.05791841 & 0.04284787 & 0 & 0 & 0 \\ 8.4957912 & -4.92206720 & -2.52848895 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \tag{4.14}$$

and

$$\underline{\sigma}_Y = \begin{bmatrix} \underline{\sigma} & \underline{0} \\ \underline{0} & \underline{0} \end{bmatrix} = \begin{bmatrix} 0.007739484 & 0 & 0 & 0 & 0 & 0 \\ 0.011560909 & 0.009628814 & 0 & 0 & 0 & 0 \\ -0.059952797 & -0.031578778 & 0.2800872 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.15}$$

As explained in Section 3.3.2, we also need the eigenvalues and corresponding eigenvectors of $e^{\underline{B}}$ to find the conditional expected value and covariance function of $\underline{Y}_t$. The eigenvalues of $e^{\underline{B}}$ are 1, 1, 1, 0.8602965, 0.5196372 and 0.1056362. The corresponding eigenvectors are

$$\begin{bmatrix} 0 & 0 & 0 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ 0 & 0 & 0 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ 0 & 0 & 0 & \gamma_{31} & \gamma_{32} & \gamma_{33} \\ 1 & 0 & 0 & \gamma_{41} & \gamma_{42} & \gamma_{43} \\ 0 & 1 & 0 & \gamma_{51} & \gamma_{52} & \gamma_{53} \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -0.0839152 & -0.1381293 & -0.0509005 \\ 0 & 0 & 0 & -0.0721419 & 0.0108002 & 0.0403456 \\ 0 & 0 & 0 & -0.1504782 & -0.6546245 & -2.2477537 \\ 1 & 0 & 0 & 0.5576569 & 0.211005 & 0.0226451 \\ 0 & 1 & 0 & 0.4794176 & -0.0164984 & -0.0179493 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{4.16}$$

4.1.4 Fitting the Global Model

In the global model, the rate of return of the combined the assets is fitted by a one-dimensional AR(1) model.

We obtained the following AR(1) process to describe the rate of return of Asset Portfolio 1 (60% of long-term bond, 30% of short-term Treasury bill and 10% of equity):

$$X_{P_1,t} - 0.0689 = 0.766(X_{P_1,0} - 0.0689) + a_{1,t} \tag{4.17}$$

where $a_{1,t} \sim N(0, 3.06e-4)$.

The AR(1) process that describes the rate of return of Asset Portfolio 2 (60% long-term bond, 10% short-term Treasury bill and 30% equity) is

$$X_{P_2,t} - 0.0738 = 0.286172(X_{P_2,0} - 0.0738) + a_{2,t} \tag{4.18}$$

where $a_{2,t} \sim N(0, 0.001784)$.

We obtained the following AR(1) process to describe the rate of return of Asset Portfolio 3 (30% long-term bond, 10% short-term Treasury bill and 60% equity)

$$X_{P_3,t} - 0.0764 = 0.1114623(X_{P_3,0} - 0.0764) + a_{3,t} \tag{4.19}$$

where $a_{3,t} \sim N(0, 0.006235)$.

The average rate of return of the asset portfolio increases with the proportion of the portfolio invested in equity. The volatility of the portfolio also increases when the portfolio has more equity. The correlation between the rate of return at time t and the rate of return at time t-1 is weaker when more equity is included in the portfolio.

These AR(1) processes can be converted to the following O-U processes since they are stationary and mean-reverting ($0 < \phi < 1$):

- the O-U process for Portfolio 1

$$X_{P_1,t} - 0.0689 = e^{-\alpha_{P_1} t}(X_{P_1,0} - 0.0689) + \sigma_{P_1} \int_0^t e^{-\alpha_{P_1}(t-s)} dW_s \tag{4.20}$$

  where α_{P_1} and σ_{P_1} in this equation equal 0.2671052 and 0.01988492, respectively.

- the O-U process for Portfolio 2

$$X_{P_2,t} - 0.0738 = e^{-\alpha_{P_2} t}(X_{P_2,t-1} - 0.0738) + \sigma_{P_2} \int_0^t e^{-\alpha_{P_2}(t-s)} dW_s \tag{4.21}$$

  where α_{P_2} and σ_{P_2} in this equation equal 1.251162 and 0.06974145, respectively.

- the O-U process for Portfolio 3

$$X_{P_3,t} - 0.0764 = e^{-\alpha_{P_3} t}(X_{P_3,0} - 0.0764) + \sigma_{P_3} \int_0^t e^{-\alpha_{P_3}(t-s)} dW_s \tag{4.22}$$

where α_{P_3} and σ_{P_3} in this equation equal 2.194069 and 0.1664489, respectively.

4.2 Comparing the Models

In Chapters 2 and 3, the expression used to calculate the first and second moments of X_t, Y_t of the asset portfolios under each model were presented in detail. By plugging the numerical values of the parameters obtained in Section 4.1 into those equations in Sections 3.1 and 3.2, the calculations for the univariate model and global model become simple and straightforward. The multivariate model can be calculated as well, although it may be more arduous and time-consuming.

4.2.1 Comparing the Instantaneous Return Rates

To better understand the movement of the rate of return of the portfolios, the conditional expected value and variance of X_t for each asset in the portfolio are studied first. By using the annual return rates of year 2007 as the starting value, $\underline{x}_0$, the conditional expected values of the instantaneous return rate ($E(X_t|X_0)$) for each asset, under the univariate and multivariate models, are shown in Figure 4.4, and the conditional variances ($Var(X_t|X_0)$) are shown in Figure 4.5.

In the first graph of Figure 4.4, the interest rate of the long-term bond needs more than 40 years to asymptotically approach its long-term mean in the univariate model, while it needs less than 20 years to approach the same long-term mean in the multivariate model. The difference is attributed to the different dependence of the long-term bond interest rate described in the two models. In the univariate model, it depends on 93.2% of the long-term interest rate last year, so it has very high correlation with the past data and reverts very slowly. In contrast, the long-term bond interest rate in the multivariate model depends on 56.5% of the long-term interest rate of last year and 35% of the short-term interest rate of last year. The interest rate of the short-term Treasury bill has a larger reverting speed coefficient ($\alpha = 0.16$) than that of the long-term bond interest rate c ($\alpha = 0.07$). Therefore, the interest rate of the long-term bond approaches its long-term stationarity more quickly

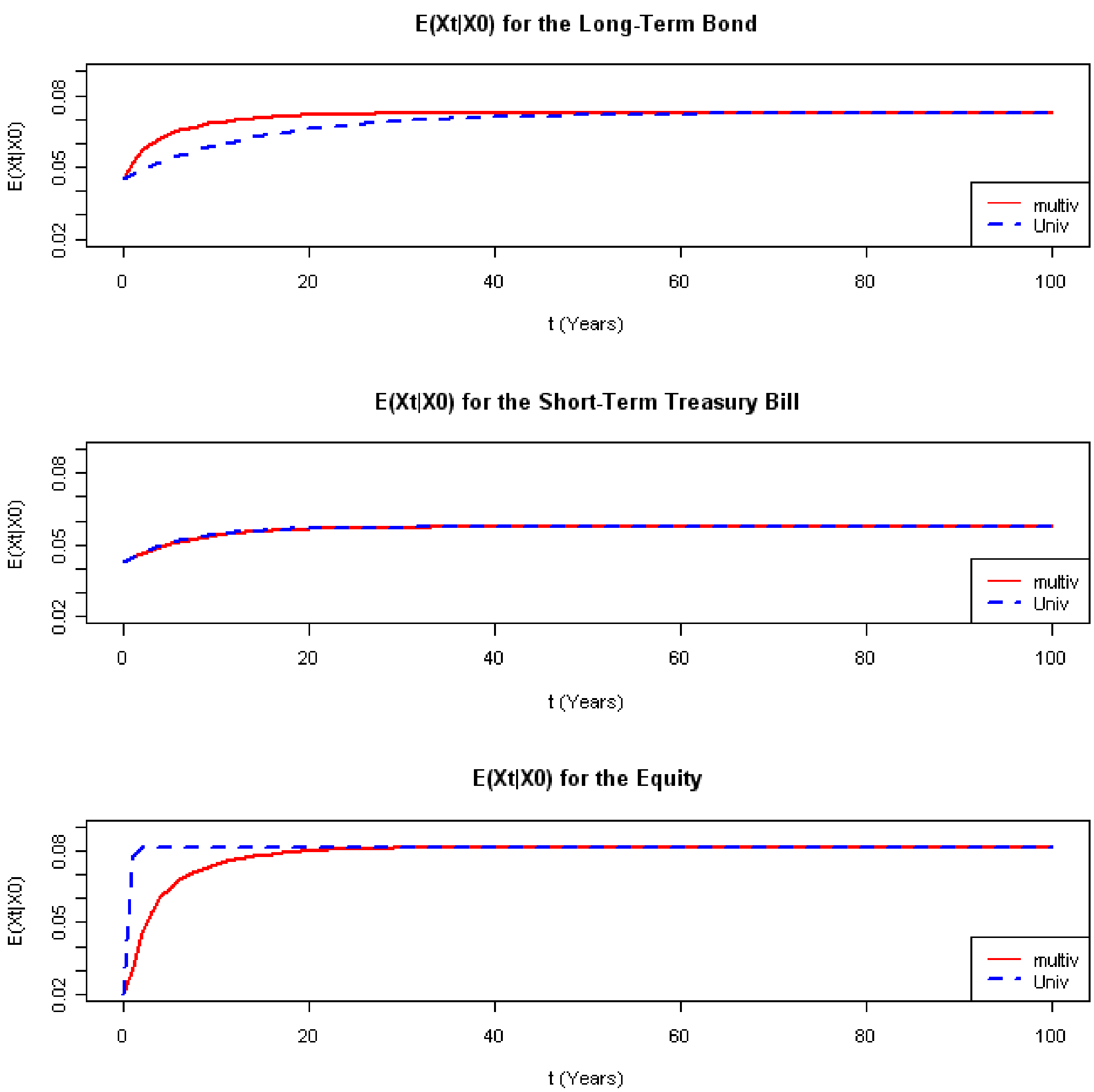

Figure 4.4: $E(X_t|X_0)$ for Each Asset

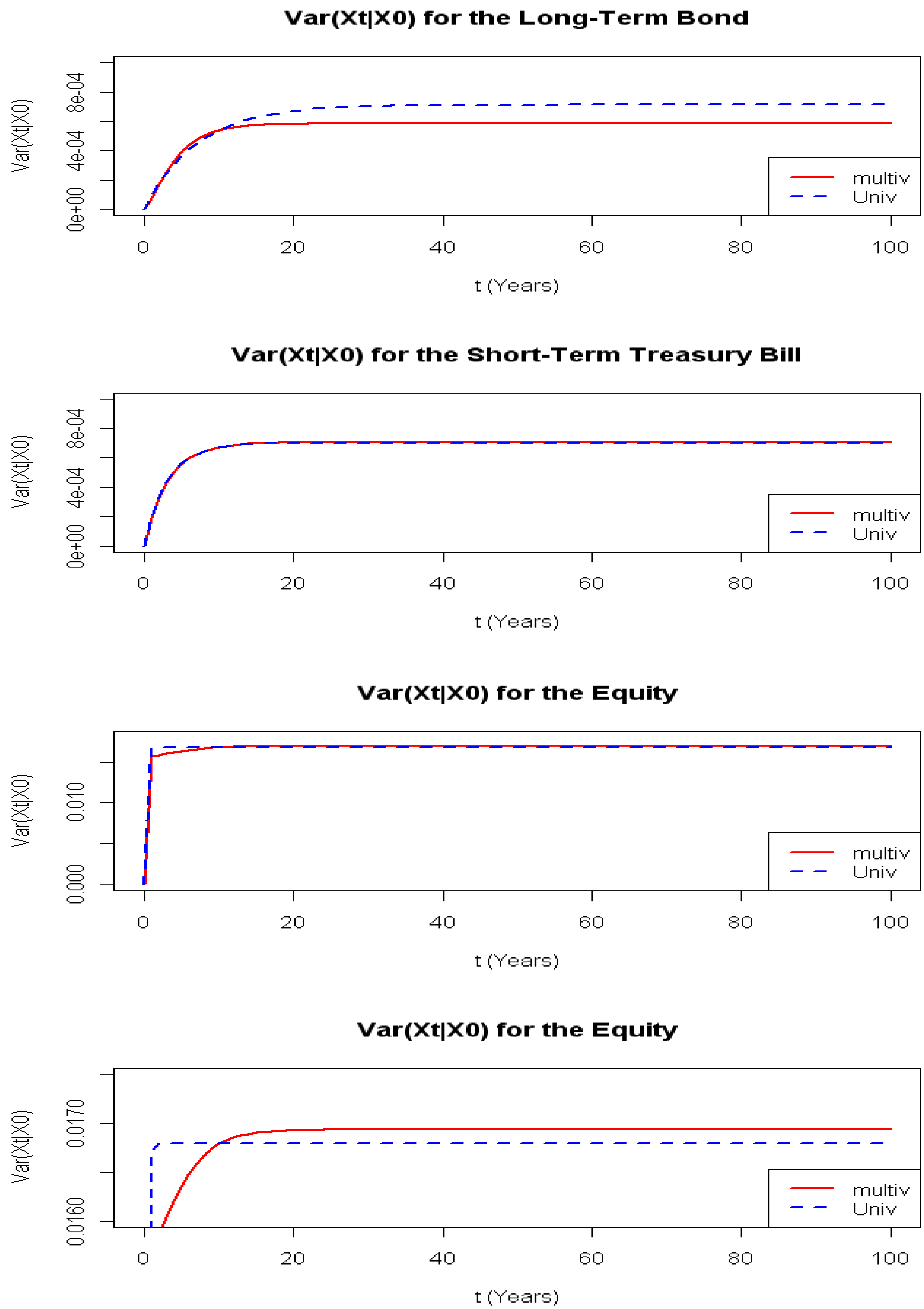

Figure 4.5: $V(X_t|X_0)$ for Each Asset

in the multivariate model. Because the interest rate of the short-term Treasury bill depends mainly on the short-term Treasury bill of last year in both the univariate and multivariate models, the conditional expected values of the short-term Treasury bill interest rate match well between the univariate model and the multivariate model shown in the second graph of Figure 4.4. Compared to the long-term bond and short-term Treasury bill, the rate of return of the equity are the most different between the univariate and multivariate models (shown in the third graph of Figure 4.4). In the univariate model, the rate of return of the equity is described as an O-U process with a large α (2.77). In the multivariate model, the rate of return of the equity is modeled as a process heavily dependent on the past interest rates of both the long-term bond and the short-term Treasury bill. Since the interest rates of both the long-term bond and the short-term Treasury bill have a slow mean reverting speed, the rate of return of the equity in the multivariate model has a slow reverting speed as well. As a result, the rate reverts to its long-term mean much more quickly in the univariate model than in the multivariate model.

The conditional variances of the instantaneous rates of return for the long-term bond and equity are different in the univariate and multivariate models as well (shown in Figure 4.5). The stationary variance of the long-term bond calculated in the multivariate model is smaller than the one calculated in the univariate model due to the smaller covariance among the series estimated by the multivariate model (shown in the first graph of Figure 4.5). In contrast, the stationary variance of the instantaneous return rate of the equity calculated in the multivariate model is larger than the one in the univariate model, because the covariance between the time series calculated from the multivariate model is larger. Additionally, the conditional variance of the rate of the equity reaches stationarity much more slowly in the multivariate model (about 15 years) than in the univariate model (around one year). Both the third and fourth graphs in Figure 4.5 display the conditional variance of the equity over time. The scale of the Y-axis in the third graph is 0-0.02 and the scale of the Y-axis in the fourth graph is 0.016-0.0175. The third graph shows that the conditional variance of X_t of the equity is much larger than the conditional variances of X_t of the long-term bond and short-term Treasury bill. The fourth graph shows the difference of X_t of the equity between the univariate and multivariate models.

Having studied and compared the conditional expected values and variances of X_t for each asset between the univariate and multivariate models, we can further work on the rates of the return of the entire asset portfolios. In the portfolios, the covariances among the assets need to be considered as well. By using the annual return rates of 2007 as the starting value, $\underline{x}_0$, the conditional expected values ($\mathrm{E}(X_t|X_0)$) of the instantaneous return rate for Portfolio 1 (60% long-term bond, 30% short-term Treasury bill and 10% equity), Portfolio 2 (60% long-term bond, 10% short-term Treasury bill and 30% equity) and Portfolio 3 (30% long-term bond, 10% short-term Treasury bill and 60% equity) are shown in Figure 4.6. The conditional variances ($\mathrm{Var}(X_t|X_0)$) of the instantaneous return rates for these portfolios are shown in Figure 4.7.

The curves of both $\mathrm{E}(X_t|X_0)$ and $\mathrm{Var}(X_t|X_0)$ are quite different among these three models in short-term (shown in Figures 4.6 and 4.7). When the proportion of equity is about 10%, the curves of $\mathrm{E}(X_t|X_0)$ and $\mathrm{Var}(X_t|X_0)$ calculated from the multivariate model are close to the curves calculated from the global model. As the percentage of equity increases, the reverting speed in the global model becomes higher than the reverting speed for the multivariate model due to the rapidly increasing of α in the global model. The conditional expected values and variances calculated from the univariate model have a jump in the first year due to the equity component. They then revert to their long-term means slowly, depending on the long-term bond and short-term Treasury bill components. Therefore, the curves calculated from the univariate model are always initially above the curves in the multivariate model, and eventually drop below the curves for the multivariate model.

For each portfolio, the conditional expected value of X_t calculated in all three models asymptotically approach the same stationary value (shown in Figure 4.6). And this stationary value increases as the proportion of equity in the portfolio increases. The conditional variances of X_t in three models approach different stationary values. In long term, the covariance among the rates of return of the assets are much greater than the variance difference of the long-term bond return rate between the univariate model and the multivariate model. Hence, in all three asset portfolios, the stationary value of $\mathrm{Var}(X_t|X_0)$ calculated from the univariate model is smaller than those calculated from the multivariate and global models. In contrast, the global model takes those correlation into consideration and can be a good and quick way to estimate the variance of X_t calculated under the multivariate model.

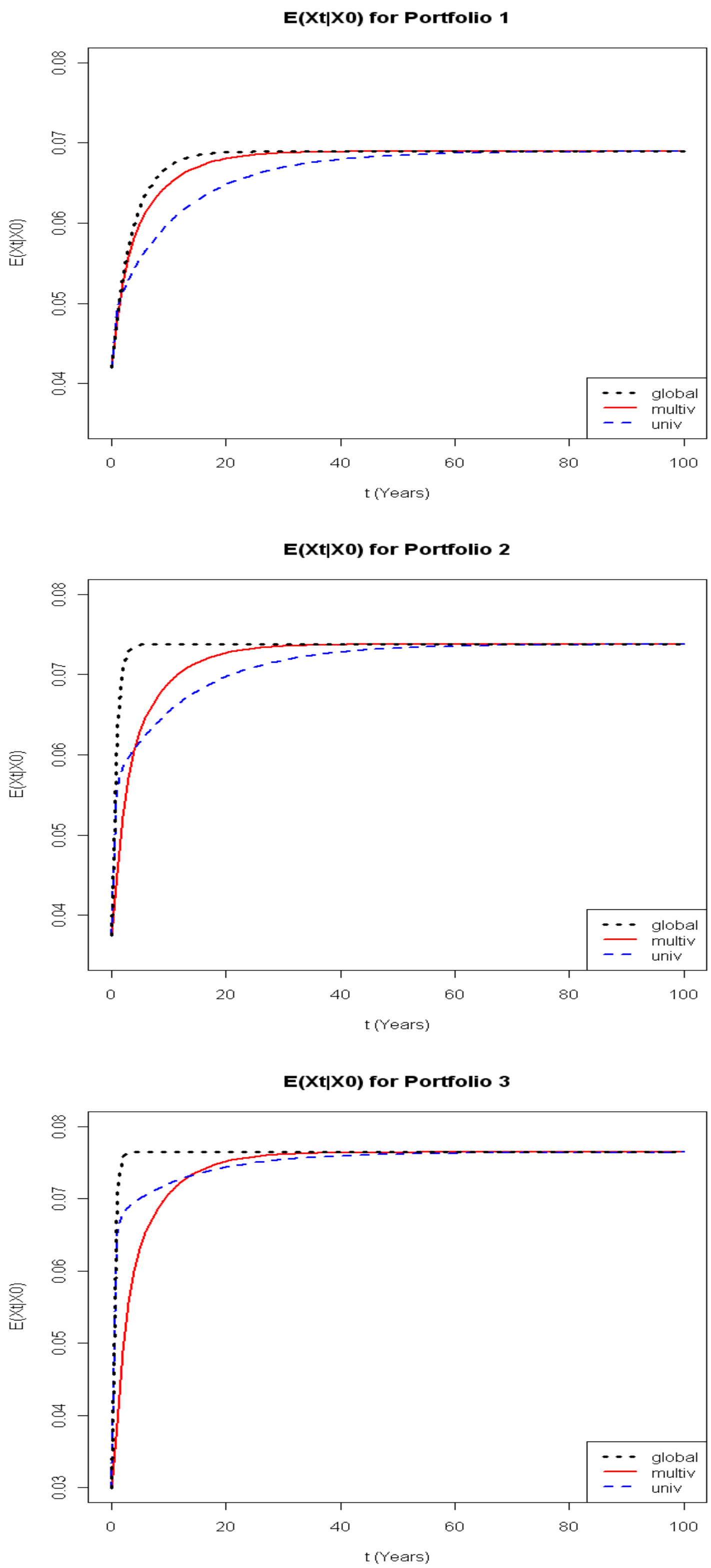

Figure 4.6: $E(X_t|X_0)$ for Each Portfolio, Starting in 2007, Using Annual Return Rates

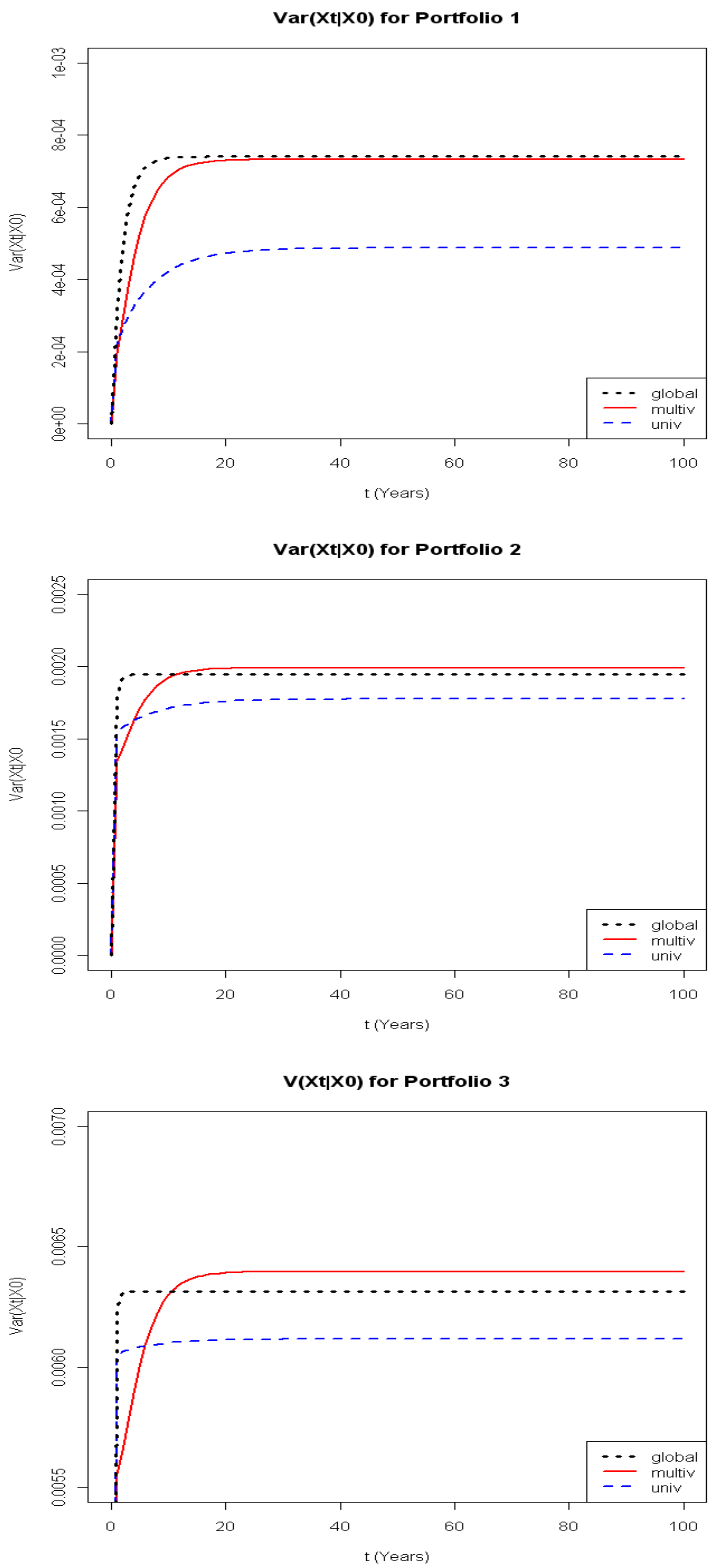

Figure 4.7: $Var(X_t|X_0)$ for Each Portfolio, Starting in 2007, Using Annual Return Rates

4.2.2 Comparing the Accumulated Return Rates

We are also interested in the accumulation function of the rate of return, since it is essential for calculating the present or future value of the portfolio. The conditional expected values of Y_t for the portfolios determined by the different models are shown in Figure 4.8.

The conditional expected values of Y_t of each portfolio calculated from the three models are very close, because the integration of the difference of $E(X_t|X_0)$ among the three models is small. The result obtained from the global model is always slightly greater than the one obtained from the univariate model due to the fact that the long-term bond and short-term Treasury bill asset components are combined with the equity in the global model and assumed to have higher mean reverting speeds from their low starting values. $\mathrm{E}(Y_t|X_0)$ increases more quickly with the proportion of equity in the portfolio in the univariate and global models than in the multivariate model. In the multivariate model, the rate of return for the equity is assumed to be dependent on the rates of return for both long-term bond and short-term Treasury bill, and reverts to its long-term mean slowly from its low starting value. In the univariate and global models, the rates of return of the equity are assumed to approach their long-term means more quickly. Hence, $\mathrm{E}(Y_t|X_0)$ is more sensitive to an increase in the proportion of equity under the univariate and global models than under the multivariate model.

The variances of Y_t ($Var(Y_t|X_0)$) of each portfolio from the three models are shown in Figure 4.9. Unlike $E(Y_t|X_0)$, $Var(Y_t|X_0)$ are quite different among the three models. Due to the accumulation of the covariance, $Var(Y_t|X_0)$ of all the portfolios calculated from the multivariate model exceed those calculated from the global and univariate models after a certain period (less than 15 years) and increase much more quickly after that. As the equity proportions increase in the portfolios, the difference caused by the covariance becomes more obvious, because the multivariate model is the only model that can fully consider the correlations between the equity and other assets. The curves for the $Var(Y_t|X_0)$ under the global model increases most quickly at the beginning, but it may be outpaced by the curves calculated from both the multivariate and univariate models in the long run. The global model includes all the variance and covariance in the assets with one univariate O-U process. Also, it underestimates the covariance and overestimates the variance for our three asset portfolios. As a result, $Var(Y_t|X_0)$ estimated from the global model is initially greater than those estimated from the other two models, and smaller in later period when more covariance terms are included in the calculation. Figure 4.10 presents the first 40-year $Var(Y_t|X_0)$ of the portfolios calculated from the different models. The curve calculated from the global model is close to the curve from the multivariate model in the first 15 years when the proportion of equity is 10%. In fact, the global curve will be closer to the curve of the multivariate model in the first 20 years when the proportion of equity is less (data not shown here).

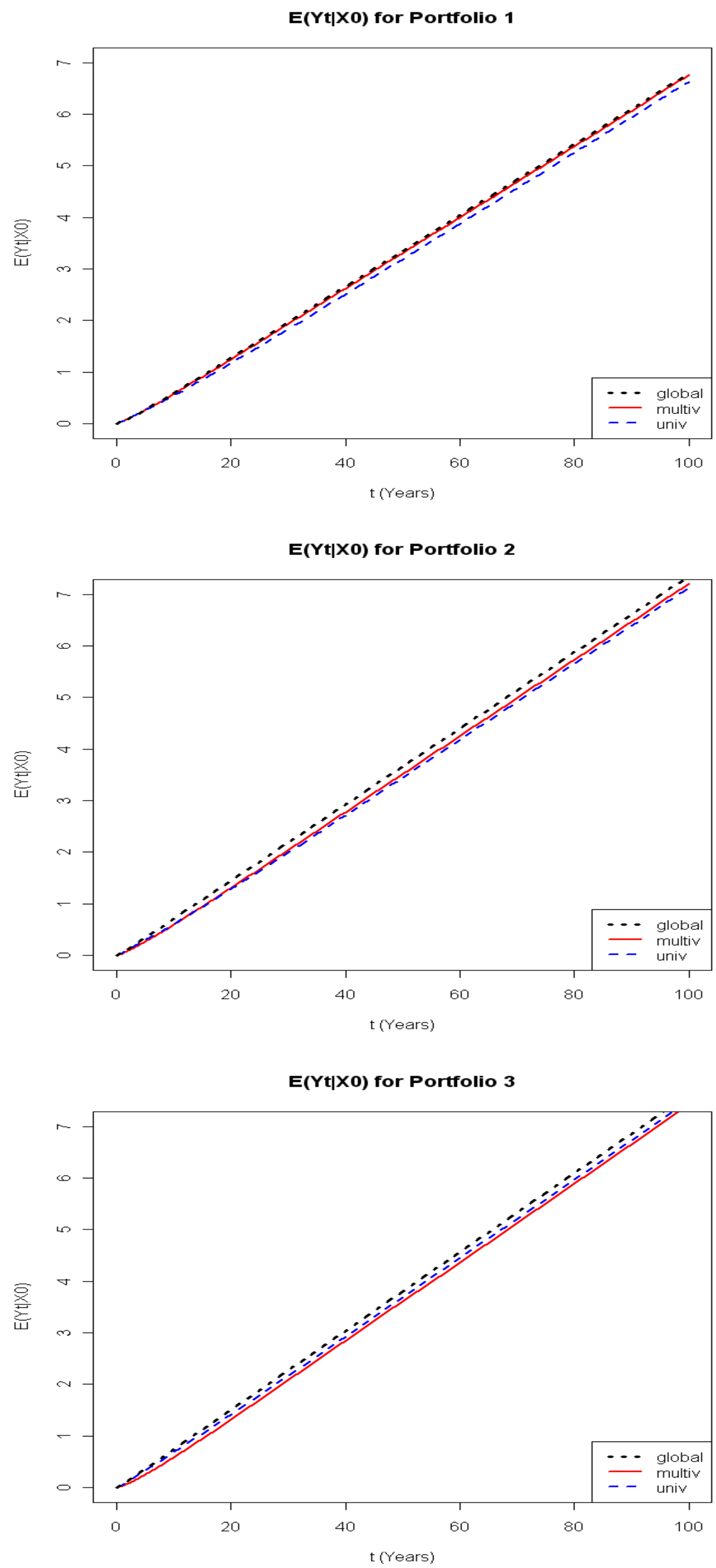

Figure 4.8: $E(Y_t|X_0)$ for Each Portfolio, Starting in 2007, Using Annual Return Rates

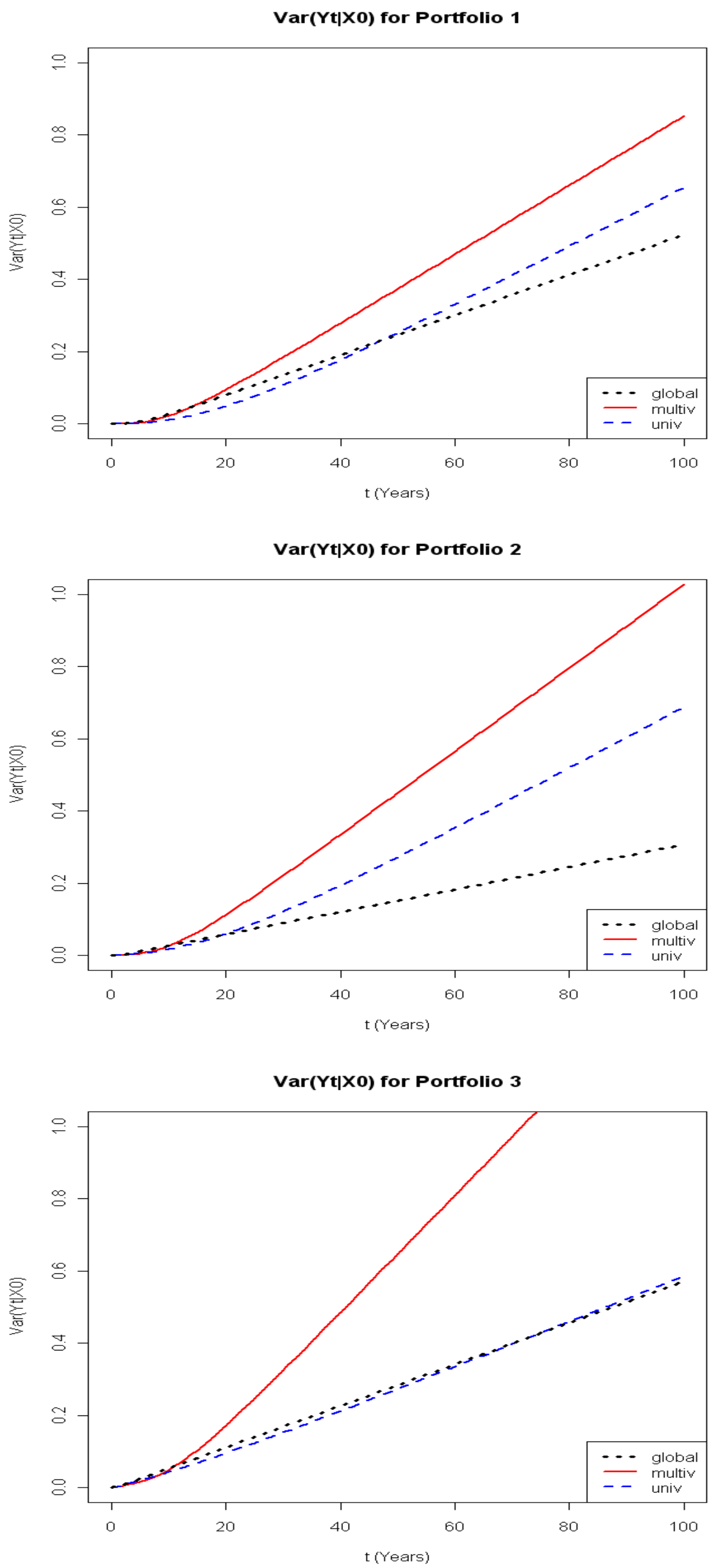

Figure 4.9: $Var(Y_t|X_0)$ for Each Portfolio, Starting in 2007, Using Annual Return Rates

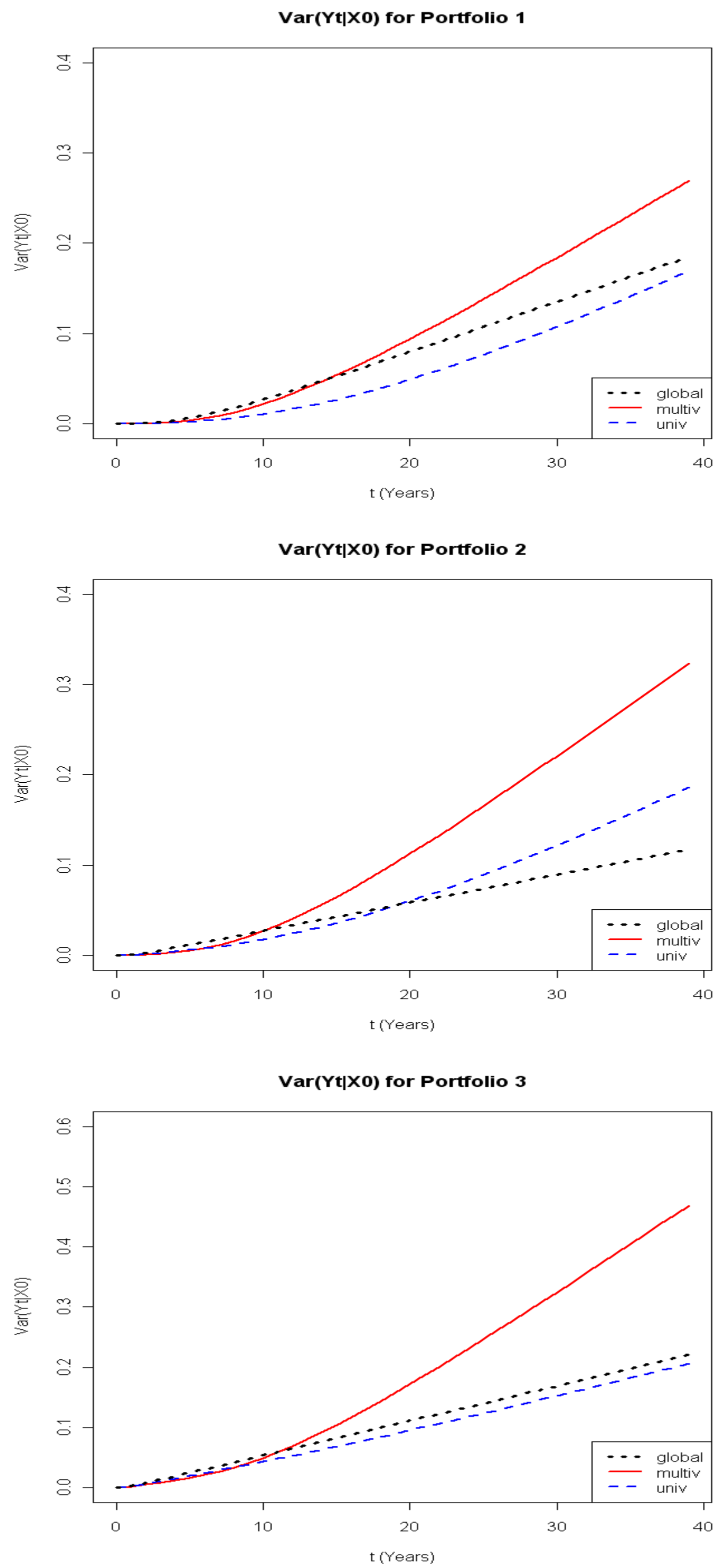

Figure 4.10: $Var(Y_t|X_0)$ for Each Portfolio for the First Forty Years, Starting in 2007, Using Annual Return Rates

Chapter 5

Applications

In Chapter 4, the parameters of the multivariate, univariate and global models were estimated from historical data for ten-year Constant Maturity Treasury Bills, three-month Treasury Bills and the S&P 500 Index. Furthermore, the conditional expected value and variance of Y_t for the portfolios were calculated using these three models. In this chapter, we will illustrate how to use the calculated results to price annuity products and optimize asset investment strategies.

5.1 Annuity Pricing

The discount factor for any future payment to be received at time t is e^{-Y_t}. Since Y_t has a normal distribution, e^{-Y_t} has a lognormal distribution. Therefore, the expected value and variance of the present value of any future payment can be obtained by using the following equations

$$E(e^{-Y_t}) = e^{-E(Y_t)+0.5Var(Y_t)}, \tag{5.1}$$

$$Var(e^{-Y_t}) = e^{-2E(Y_t)+Var(Y_t)}(e^{Var(Y_t)} - 1) \tag{5.2}$$

and

$$Cov(e^{-Y_t}, e^{-Y_s}) = e^{-E(Y_t)-E(Y_s)+0.5[Var(Y_t)+Var(Y_s)]}(e^{Cov(Y_t,Y_s)} - 1) \tag{5.3}$$

where $E(Y_t)$, $Var(Y_t)$ and $Cov(Y_t, Y_s)$ were derived in Chapters 3 and 4.

5.1.1 Pricing n-Year Certain Annuity-Immediate Products

An n-year certain annuity-immediate product is an insurance product that does not involve mortality risk. Only the interest risk needs to be considered in the product pricing. The present values of the future payments (Z) of an n-year certain annuity-immediate can be expressed as

$$Z = \sum_{i=1}^{n} e^{-Y_i} \tag{5.4}$$

where Y_i is the accumulated interest rate at time i. Therefore, we have

$$E(Z) = \sum_{i=1}^{n} E(e^{-Y_i}) = \sum_{i=1}^{n} e^{-E(Y_i)+0.5Var(Y_i)}, \tag{5.5}$$

$$\begin{aligned} E(Z^2) &= \sum_{i=1}^{n}\sum_{j=1}^{n} E(e^{-Y_i-Y_j}) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} e^{-E(Y_i)-E(Y_j)+0.5[Var(Y_i)+Var(Y_j)+2Cov(Y_i,Y_j)]} \end{aligned} \tag{5.6}$$

$$\tag{5.7}$$

and

$$Var(Z) = E(Z^2) - [E(Z)]^2. \tag{5.8}$$

E(Z), the net single premium of the certain annuity-immediate, is normally denoted by $a_{\overline{n}|}$. To reduce computation time, E(Z) and E(Z^2) can also be obtained from the following recursive equations

$$E_n(Z) = E_{n-1}(Z) + e^{-E(Y_n)+0.5Var(Y_n)} \tag{5.9}$$

and

$$E_n(Z^2) = E_{n-1}(Z^2) + 2\sum_{i=1}^{n-1} e^{-E(Y_i)-E(Y_n)+0.5[Var(Y_i)+Var(Y_n)+2Cov(Y_i,Y_n)]} + e^{-2E(Y_n)+2Var(Y_n)} \tag{5.10}$$

where $E_1(Z) = e^{-E(Y_1)+0.5Var(Y_1)}$ and $E_1(Z^2) = e^{-2E(Y_1)+2Var(Y_1)}$. The expected values, standard deviations and coefficients of variation of Z calculated with the three models are summarized in Table 5.1. This table illustrates some salient properties of the three models in pricing n-year certain annuity. First, the net single premiums (E(Z)) of a certain annuity calculated from the univariate and global models always decrease as the proportion of equity

Table 5.1: Mean, Standard Deviation and Coefficient of Variation of Z of an n-Year Certain Annuity-Immediate

		Portfolio 1[1]			Portfolio 2[2]			Portfolio 3[3]		
	n	Mlt[4]	Uni[5]	Glb[6]	Mlt	Uni	Glb	Mlt	Uni	Glb
$E(Z)$	1	0.9561	0.9542	0.9558	0.9600	0.9512	0.9487	0.9680	0.9479	0.9452
$sd(Z)$[7]	1	0.00773	0.00845	0.00995	0.0210	0.0225	0.0251	0.0435	0.0445	0.0462
$CV(Z)$[8]	1	0.00808	0.00886	0.0104	0.0219	0.0237	0.0265	0.0450	0.0469	0.0489
$E(Z)$	5	4.309	4.312	4.303	4.327	4.248	4.149	4.396	4.171	4.108
$sd(Z)$	5	0.123	0.105	0.180	0.179	0.196	0.262	0.339	0.360	0.392
$CV(Z)$	5	0.0286	0.0243	0.0417	0.0414	0.0462	0.0631	0.0770	0.0864	0.0955
$E(Z)$	10	7.524	7.586	7.491	7.516	7.382	7.047	7.655	7.144	6.954
$sd(Z)$	10	0.460	0.346	0.570	0.533	0.499	0.624	0.838	0.826	0.890
$CV(Z)$	10	0.0612	0.0456	0.0760	0.0710	0.0676	0.0886	0.109	0.116	0.128
$E(Z)$	25	12.84	13.14	12.65	12.62	12.47	11.45	12.86	11.74	11.23
$sd(Z)$	25	1.73	1.29	1.70	1.83	1.43	1.42	2.45	1.89	1.96
$CV(Z)$	25	0.135	0.0983	0.135	0.145	0.115	0.124	0.190	0.161	0.175

[1] Portfolio 1 = 60% long-term bond, 30% short-term Treasury bill and 10% equity
[2] Portfolio 2 = 60% long-term bond, 10% short-term Treasury bill and 30% equity
[3] Portfolio 3 = 30% long-term bond, 10% short-term Treasury bill and 60% equity
[4] Mlt = Multivariate Model
[5] Uni = Univariate Model
[6] Glb = Global Model
[7] sd(Z) = standard deviation of Z
[8] CV(Z) = coefficient of variation of Z

in the portfolio increases. However, E(Z) calculated from the multivariate model can go either direction as the proportion of equity increases. Under the univariate and global models, the rates of return of the equity asymptotically converge to their long-term means quickly from their low starting values. In contrast, in the multivariate model, the return rate for the equity reverts to its long-term mean slowly due to its dependence on the rates of return of both long-term bond and short-term Treasury bill. Therefore, the increment of $E(Y_t|X_0)$ calculated from the univariate and global models is greater than the one calculated from the multivariate model for the one-unit increase of the proportion of equity. Additionally, in the univariate and global models, the increment of $Var(Y_t|X_0)$ caused by increasing the proportion of equity is smaller due to the underestimation of the variance of the equity and the covariance of the equity with other assets. Therefore, under the univariate and global models, the increment of $E(Y_t|X_0)$ is always greater than the increment of the $Var(Y_t|X_0)$ caused as more equity components are added in the portfolio. As a result, E(Z) calculated from these two models decreases as the proportion of equity increases. In the multivariate model, the increment of $E(Y_t|X_0)$ may not counteract the increment of the $Var(Y_t|X_0)$, so E(Z) may even increase when more equity is included in the portfolio. Secondly, as n increases in the annuity contract, the standard deviation of Z (sd(Z)) calculated from the multivariate model increases more quickly than those calculated from the univariate and global models. This is because the multivariate model considers more covariance among the assets, which becomes larger as the payment duration increases. Thirdly, for an annuity with a short payment period, Var(Z) calculated from the global model is greater than Var(Z) calculated from the other two models. With the elongation of the payment period, Var(Z) calculated from the global model is exceeded by the one calculated from the multivariate model relatively more quickly (compared to the univariate model). Var(Z) calculated from the univariate model approaches Var(Z) calculated from the global model at a lower speed, which can be reduced further by increasing the proportion of equity in the portfolio. This can be explained by the behavior of $Var(Y_t|X_0)$ (Figure 4.10). The underestimation of the covariance and overestimation of the variance in the global model results in a larger $Var(Y_t|X_0)$ over a short period, but smaller $Var(Y_t|X_0)$ over a long period.

In fact, all the results in Table 5.1 can be reasonably explained by Figures 4.8 and 4.9 in Chapter 4, which illustrate $E(Y_t|X_0)$ and $Var(Y_t|X_0)$, because E(Z) and Var(Z) are calculated based on conditional expected value, variance and covariance of Y_t.

5.1.2 Pricing Whole-Life Annuity Products

Knowing E(Z) and Var(Z) for an n-year certain annuity-immediate, we can calculate the expected value and variance of the present value of future benefit payments for a whole-life annuity-immediate product by using the following equations

$$E(Z) = a_x = \sum_{k=1}^{\omega-x-1} {}_kp_x e^{-Y_k} = \sum_{k=1}^{\omega-x-1} {}_{k|}q_x E(a_{\overline{k|}}), \tag{5.11}$$

$$E(Z^2) = E(E(Z^2|K)] = \sum_{k=1}^{\omega-x-1} {}_{k|}q_x E(a_{\overline{k|}}{}^2) \tag{5.12}$$

and

$$Var(Z) = E(Z^2) - [E(Z)]^2, \tag{5.13}$$

where $E(a_{\overline{k|}})$ denotes the expected value of the present value of future benefit payments of a k-year certain annuity-immediate. $E(a_{\overline{k|}}{}^2)$ denotes the second moment of the present value of future benefit payments of a k-year certain annuity-immediate. Mortality Table UP-94 is used for the calculation.

Figure 5.1 illustrates the expected value of Z of whole-life annuities for males over ages. In Figure 5.1, E(Z) calculated from all three models always decreases as the age at issue increases. For a whole-life annuity, increasing the age at issue usually increases the probability of early death during the contract (with some exceptions between ages 20 and 30), but also shortens every contract duration. These two effects make the expected value of Z decrease as the age at issue increases. Due to the decreasing of E(Z), the difference between the E(Z) values calculated from the three models also decreases as the age at issue increases. Similar to the net single premium of an n-year certain annuity-immediate, E(Z) calculated from the multivariate model is much less sensitive to increases in the proportion of equity in the portfolio than that of the other two models. Under the univariate and global models, E(Z) can be reduced quickly by increasing the proportion of equity. On the contrary, E(Z) calculated from the multivariate model decreases slowly or may even increase (depending on the payment period and the proportions of the three asset types in the portfolio) as the proportion of equity increases.

The variance of Z for whole-life annuities for a male is displayed over different ages

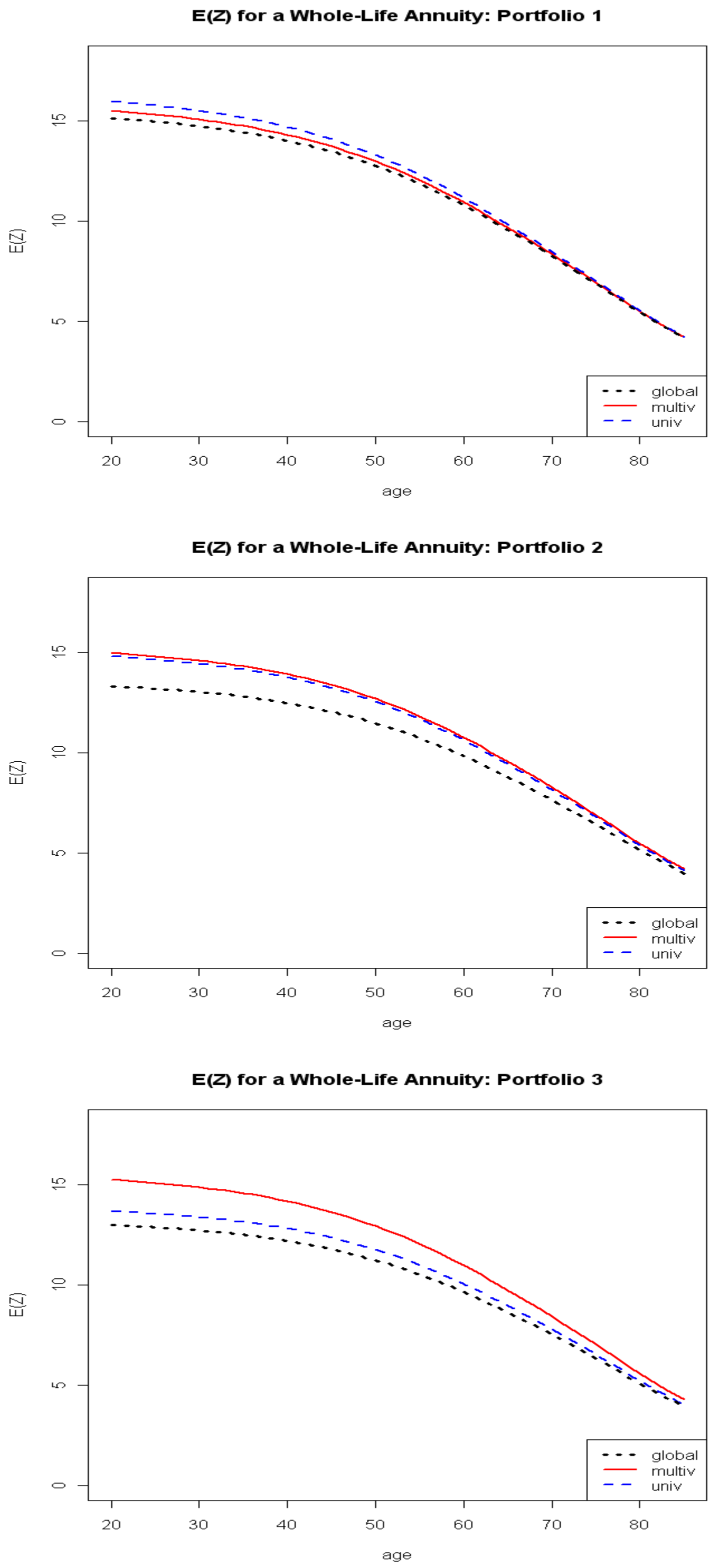

Figure 5.1: E(Z) for a Whole-Life Annuity

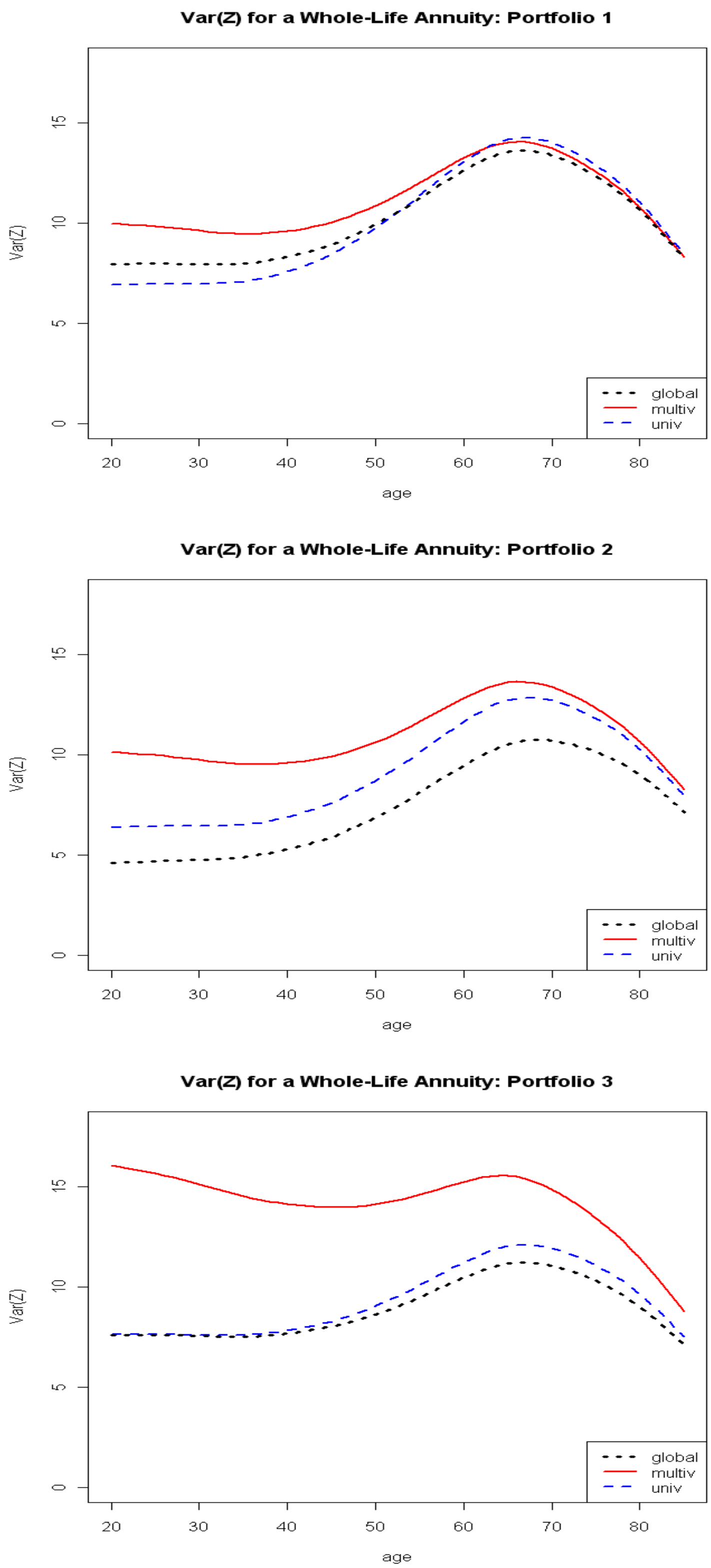

Figure 5.2: Var(Z) for a Whole-Life Annuity

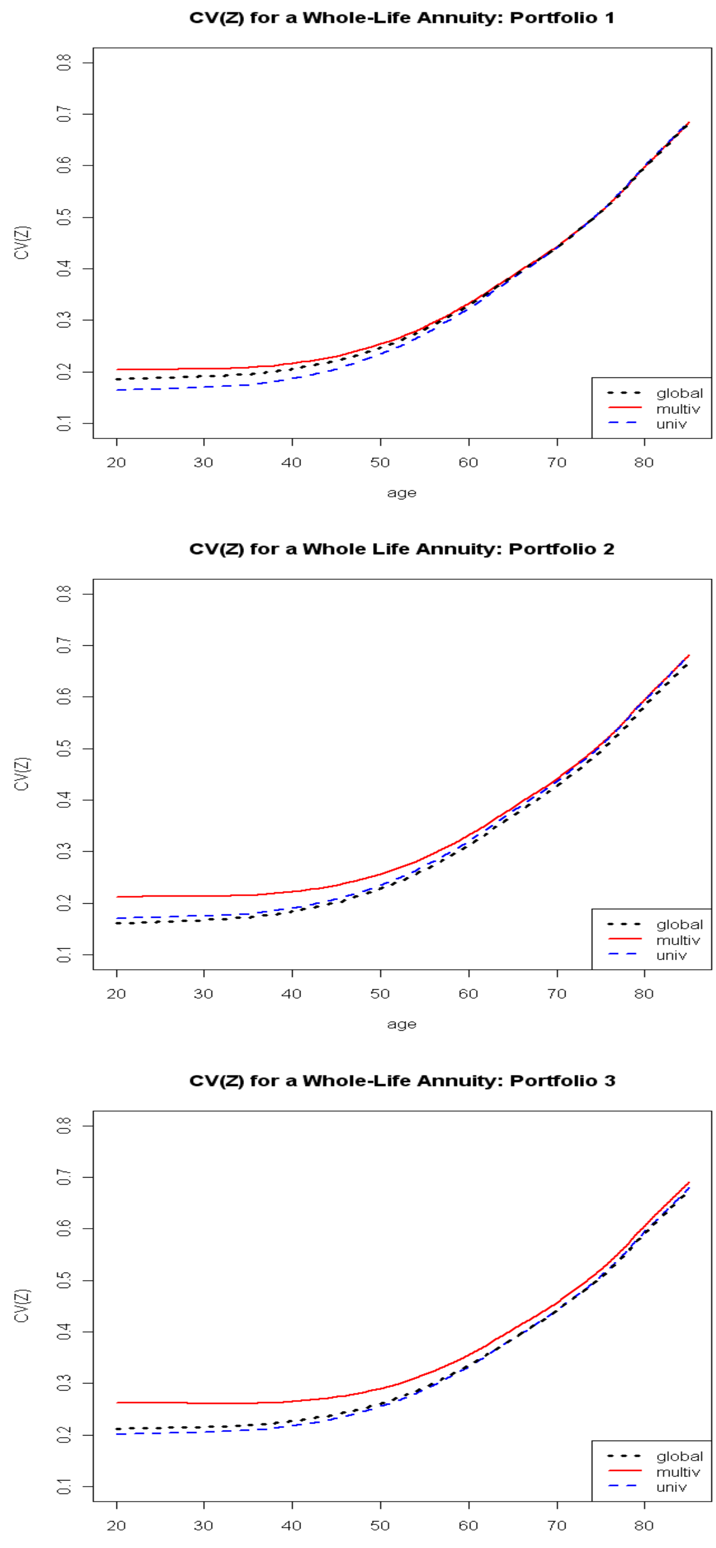

Figure 5.3: CV(Z) for a Whole-Life Annuity

in Figure 5.2. The interest risk always decreases as the age at issue increases, due to fewer payments and a shorter payment period. Since most death occurs between ages 70 and 90, the mortality risk increases as the age at issue approaches 70 and quickly decreases after that. Therefore, the graph of Var(Z) for each of the three models has a hump around age 70 when the mortality risk is the main component of Var(Z). For the univariate and global models, the uncertainty due to mortality is much greater than the uncertainty due to the force of interest. For this reason, the shapes of Var(Z) tend to be given by the uncertainty due to mortality under the univariate and global models. Because the multivariate model gives more consideration to the autocovariance and covariance of equity, the investment risk calculated from the multivariate model is larger than that calculated from the other models when the age at issue is small. In the multivariate model, the largest component of Var(Z) is due to the interest risk when the age at issue is small. So the curve of Var(Z) calculated from the multivariate model decreases when the age at issue is small. When the age at issue is greater than 50, the mortality risk becomes more influential and determines the shape of Var(Z). As a result, there is a large difference in the Var(Z) between the multivariate model and the other two models when the age at issue is small. The difference becomes much more significant when the proportion of equity is high (as shown in Asset Portfolio 3).

As for the coefficient of variation of Z, in all three models, it increases rapidly with the age at issue when the age at issue is large and more slowly when the age at issue is small (shown in Figure 5.3). For a fixed age at issue, it also increases more quickly with the proportion of equity under the multivariate model than the other two models.

5.2 Optimal Asset Allocation Strategy

If the asset proportions in a portfolio are known and fixed, the actuarial present value of future annuity payment can be estimated from the three models. In this section, we assume that the asset proportions in the portfolio are undetermined and need to be optimized to maximize the profit for a whole-life policy sold to a 65-year-old male. For simplicity, we assume that only three kinds of assets are available in the market: ten-year long-term bonds, three-month short-term Treasury bills and equity. Additionally, the three assets are re-balanced frequently and their proportions in the portfolio remain equal to their starting values. The return rates of portfolios in year 2007 are assumed to be the starting values.

The asset portfolio which has the lowest net single premium can yield the maximum average profit if other expenses are ignored. Intuitively, the asset allocation with the lowest net single premium can be considered as one of the optimal asset allocation strategies. E(Z) for investment portfolios composed of three assets with various proportions are calculated and shown in Figure 5.4. In these three-dimensional graphs, the proportions invested in the long-term bond and short-term Treasury bill are displayed on the two horizontal axes and E(Z) is displayed on the vertical axis. The rest of the portfolio is invested in equity, which is not displayed directly on the graphs. The sum of the proportions invested in the three assets is always 100%. Area where sum of proportion invested in long -term bond and short-term Treasury bill would exceed 100% are left blank in the graphs. Consistent with ealier results, when the proportion of equity increases, E(Z) decreases quickly in the univariate and global models, but not in the multivariate model (shown as black surfaces in the graphs). Therefore, as summarized in Table 5.2, the minimum E(Z) calculated from the univariate and global models occurs with 100% asset allocation in the equity. However, E(Z) calculated from the multivariate model reaches its minimal value when the portfolio has 100% allocation in the long-term bond. As mentioned before, the univariate and global models underestimate the variance and covariance of the equity in the asset portfolio and overestimate the speed at which the rate of return of the equity reverts to its long-term mean from the low starting value. As a result, the univariate and global models yield different optimal asset allocation strategy from the multivariate model. Based on the net single premium criteria, the multivariate model suggests investing 100% of assets in long-term bonds while the other two models suggest investing 100% of assets in equities. Note that our conclusions for all three models are based on the return rate starting at the 2007 value which is below the long-term average. If the starting value is changed, the models may make completely different suggestions. For example, assuming that the starting return rates of the asset portfolios are twice as high as the long-term mean, we re-calculated E(Z) under the three models. These results are shown in Figure 5.5. With a higher starting value, the multivariate model suggests investing 100% of assets in equities while the other two models suggest investing 100% of assets in long-term bonds.

Reducing risk is another criteria that can be based to optimize the asset allocation strategy.

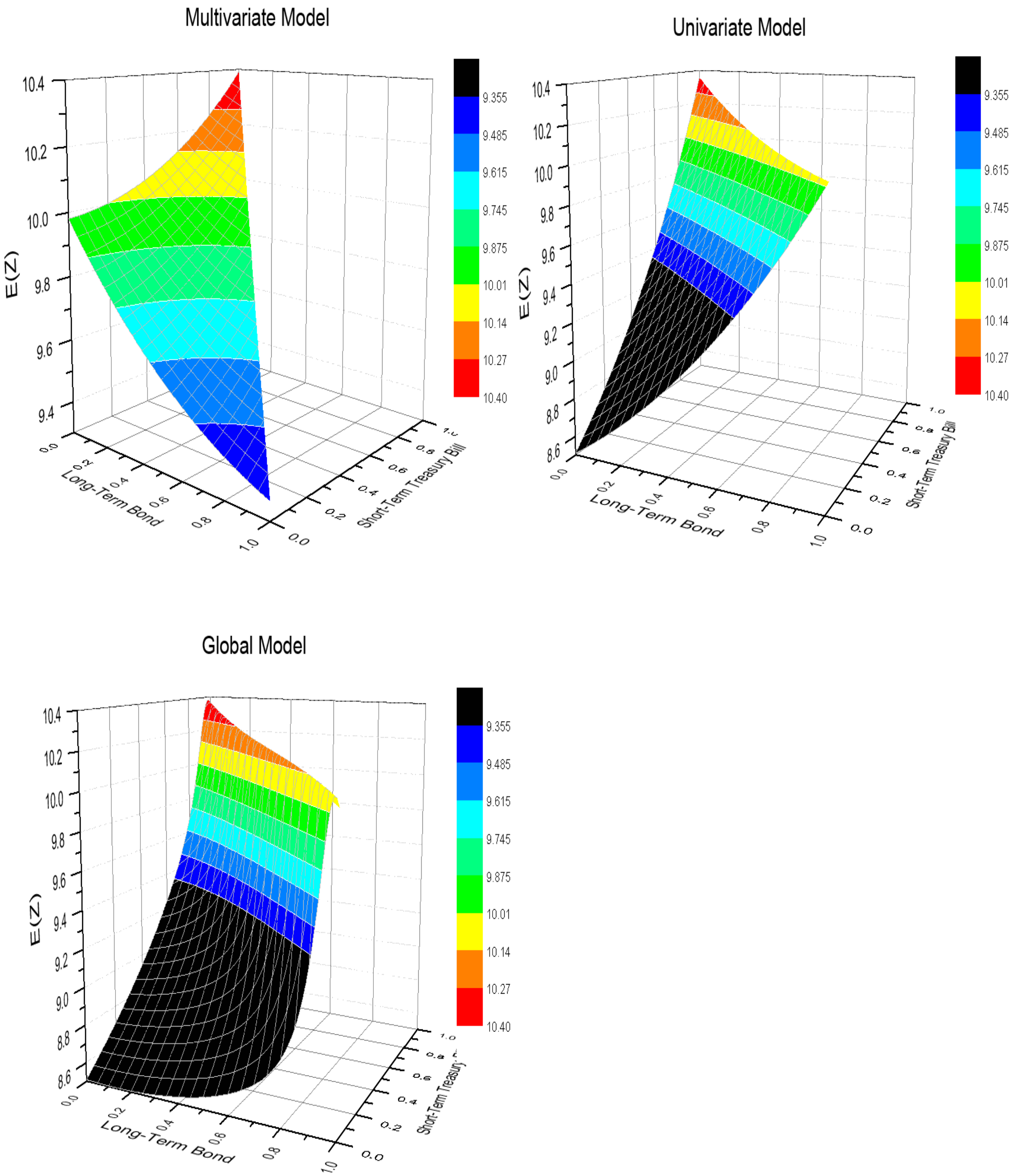

Figure 5.4: E(Z) for a Whole-Life Annuity Sold to Male, Age 65, Starting with Rate of Return in 2007

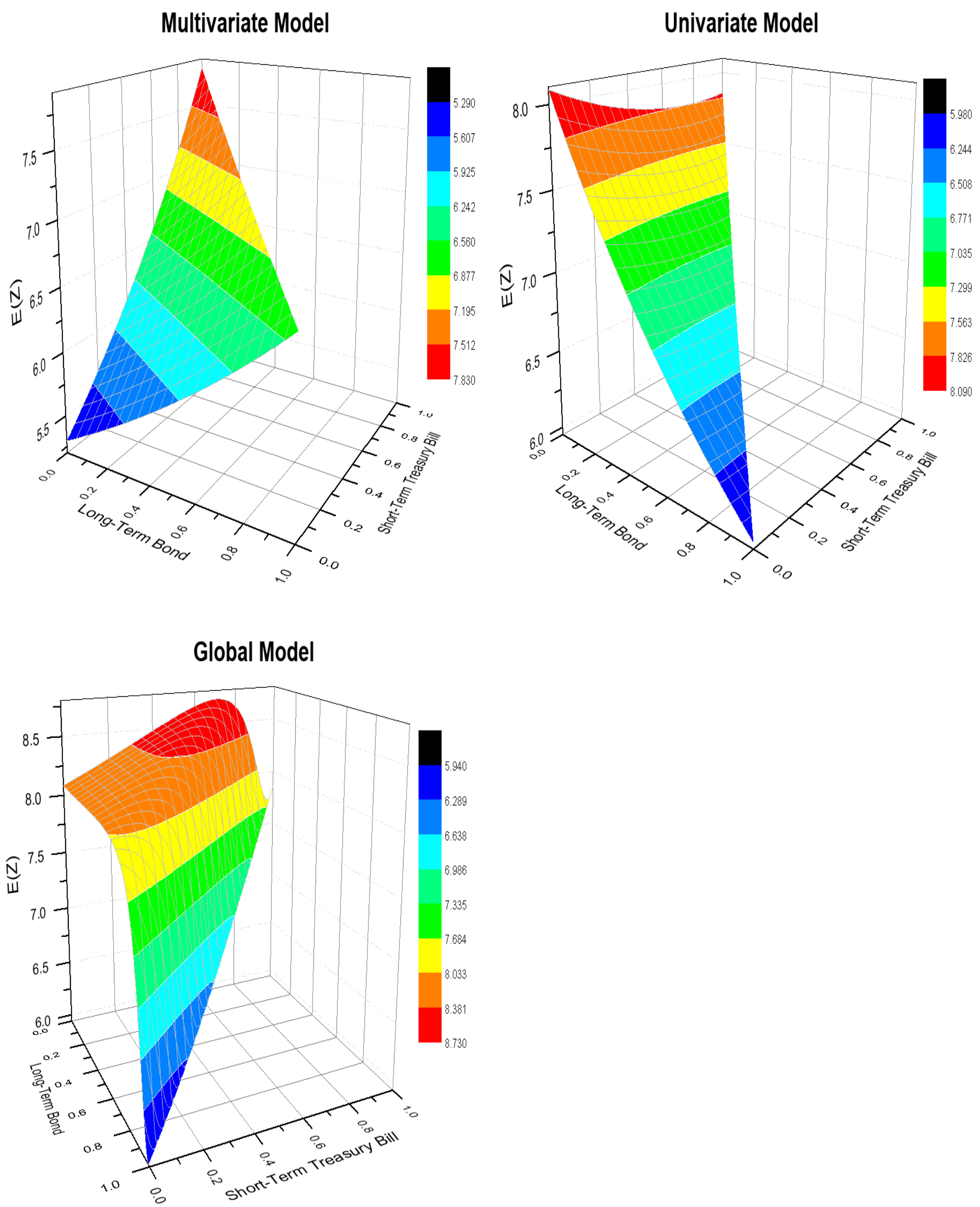

Figure 5.5: E(Z) for a Whole-Life Annuity Sold to Male, Age 65, Starting with High Return Rate

Var(Z) for investment portfolios of three assets with various proportions are shown in Figure 5.6. In the graphs, Var(Z) calculated from the multivariate model reaches its minimal value when the portfolio is invested 100% in long-term bonds. Var(Z) calculated from the univariate model reaches its minimal value when the portfolio is invested 37% in long-term bonds, 4% in short-term Treasury bills and 59% in equities. The optimal asset allocation strategy calculated from the global model to minimize the Var(Z) lies between these two models: the global model shows that the portfolio with minimal Var(Z) is invested 69% in long-term bonds and 31% in short-term Treasury bills (summarized in Table 5.2). The behavior of Var(Z) can be explained by the same reasons used to explain the behavior of E(Z).

Table 5.2: Asset Portfolios with Minimum and Maximum E(Z) and Var(Z), Starting with Rate of Return in 2007

	E(Z)		Var(Z)	
Models	Minimum	Maximum	Minimum	Maximum
Mlt	100% Long-Term[1]	100% Short-Term [2]	100% Long-Term	100% Equity
Uni	100% Equity	100% Short-Term	37% Long-Term 4% Short-Term 59% Equity	100% Short-Term
Glb	100% Equity	98% Short-Term 2% Equity	69% Long-Term 31% Equity	100% Short-Term

[1] Long-Term = Long-Term Bond
[2] Short-Term = Short-Term Treasury Bill

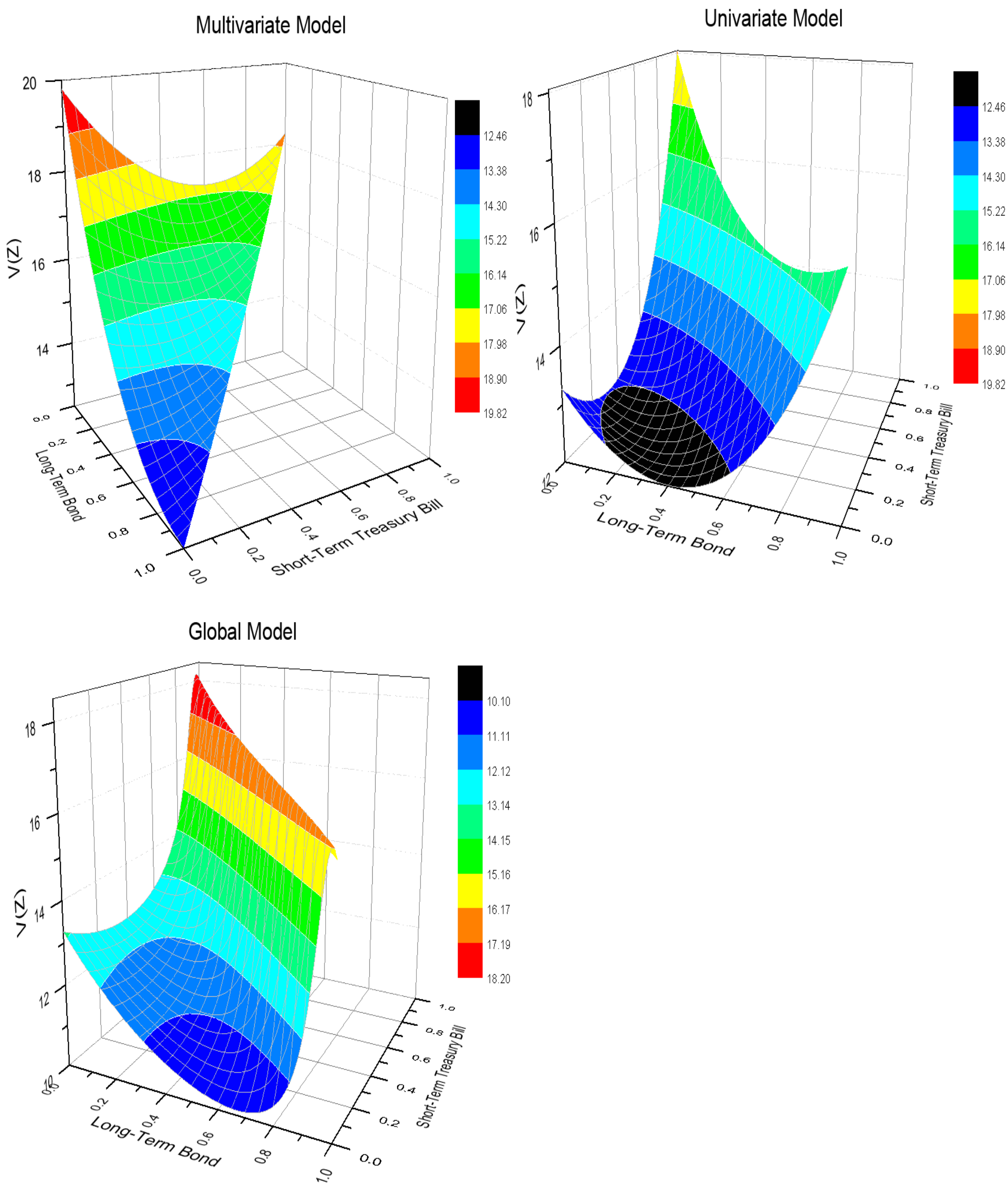

Figure 5.6: Var(Z) for a Whole-Life Annuity Sold to Male, Age 65, Starting with Rate of Return in 2007

Chapter 6

Conclusions

In this project, rates of return of asset portfolios are modeled by three different models: the univariate model, the global model and the multivariate model. Furthermore, a deterministic method is developed in our project for computing the first and second moments of the accumulation function of the return rates of the asset portfolio from the multivariate model. Therefore, the conditional expected values, variances and covariances of the instantaneous and accumulated return rates for different asset portfolios are calculated for the three models. Additionally, the expected value and variance of the net single premium for certain types of annuities are calculated. All the results are compared among the three models. With more parameters, the multivariate model takes full consideration of the correlation between the assets in one portfolio and provides a more precise estimate of the rates of return of the asset portfolios. The univariate model completely ignores the correlation between the assets in one portfolio which may lead to incorrect estimate of the return rates of the asset portfolios. In the global model, although the variances and covariances information are included, the estimate of the rates of return of the portfolio is imprecise due to the limited parameters used in the model. In most cases, the difference between the multivariate model and the other two models will have a huge impact on the estimate of the rates of return of the asset portfolio. Therefore, in practice, the multivariate model should be applied in order to obtain a more appropriate evaluation of the rates of investment return.

6.1 Comparing the Models

In our project, the asset portfolios are assumed to include three assets: the long-term bond, the short-term Treasury bill and the equity. In the univariate model, the current interest rate of the long-term bond is highly dependent on the interest rate of the previous year. In the multivariate model, the current interest rate of the long-term bond depends on 56.5% of the long-term bond interest rate of the previous year and 35% of the short-term Treasury bill interest rate of the previous year. The interest rate of the short-term Treasury bill depends mainly on the short-term Treasury bill of last year in both the univariate and multivariate models. The greatest difference between the univariate and multivariate models lies in their descriptions of the dependence of equity. The equity is described as a process close to white noise in the univariate model, while it greatly depends on the past interest rates of both the long-term bond and the short-term Treasury bill in the multivariate model. The global model combines all three assets and describes the rates of investment return as a one-dimensional O-U process in which it is difficult to identify the dependence of each individual asset of the portfolio. As the proportion of equity in the portfolio increases, the estimate from the global model becomes more like to that from the univariate model.

Because the three models describe the dependence of the assets in quite different ways, they give different estimates of the rates of return of the asset portfolio. First, $\mathrm{E}(Y_t|X_0)$, the conditional expected value of the accumulated return rate of the portfolio, is more sensitive to an increase in the proportion of equity under the univariate and global models than under the multivariate model. As the starting value is lower than the long-term mean, $\mathrm{E}(Y_t|X_0)$ increases more quickly with the proportion of equity under the univariate and global models than under the multivariate model due to the higher mean-reverting speed of the equity in the univariate and global models. For each asset portfolio, due to large covariances, $\mathrm{Var}(Y_t|X_0)$ calculated from the multivariate model exceeds that calculated from the global and univariate models after a certain period (less than 15 years) and increases much more quickly after that. The global model includes all the variance and covariance in the assets with one univariate O-U process. In our three asset portfolios, the global model underestimates the covariance and overestimates the variance. As a result, $\mathrm{Var}(Y_t|X_0)$ estimated from the global model is initially greater than that estimated from the other two models, and smaller in the long-term when more covariance needs to be included in the calculation.

6.2 Annuity Pricing and Asset Allocation Optimization

In application, the different models lead to different conclusions as well. The univariate and global models suggest the net single premium of the annuity product can be greatly reduced by increasing the proportion of equity in the asset portfolio if the starting return rate is lower than its long-term mean. The multivariate model suggests the net single premium can go either direction as the proportion of equity in the asset portfolio increases. Additionally, the multivariate model indicates the variance of Z increases rapidly with the proportion of equity, while the univariate and global models indicate Var(Z) does not always increase with the proportion of equity.

The multivariate model supports a totally different optimal asset allocation strategy for a whole-life annuity from that supported by the univariate and global models. When the starting return rate is lower, the multivariate model suggests investing 100% of the asset in long-term bonds to get a minimum net single premium. In contrast, both the univariate and global models suggest investing 100% of the asset in equities.

6.3 Future Work

There are many opportunities for future work. For instance, our project focuses on one single policy. In the future, we could expand our application of the investment models to a portfolio of policies that share a global investment asset portfolio. To be more ambitious, we could further analyze the portfolio of policies containing groups of policies with different asset portfolios. Here, the asset portfolio for each group of policies may include different proportions of assets. In this project, we assume the mortality rate is deterministic, but this may not be the case in reality. So it would also be interesting to combine the stochastic interest rate and stochastic mortality rate in future work. Finally, we use ten-year Constant Maturity Treasury Bills, three-month Treasury Bills and the S&P 500 Index to represent three assets in the asset portfolio. Researchers may replace these with other mutual funds or commonly used investment instruments in the insurance industry and then analyze the results.

Bibliography

Beekman, J.A. and Fuelling, C.P. (1990). Interest and Mortality Randomness in Some Annuities. *Insurance : Mathematics and Economics* 9, 185-196.

Beekman, J.A. and Fuelling, C.P. (1991). Extra Randomness in Certain Annuity Models. *Insurance: Mathematics and Economics* 10, 275-287.

Bellhouse, D.R., and Panjer, H. H. (1981). Stochastic Modelling of Interest Rates With Applications To Life Contingencies - Part II. *Journal of Risk and Insurance*, 48 (4): 628-637.

Boyle, P.P. (1976). Rates of Return as Random Variables. *Insurance: Mathematics and Economics*, 33: 29-47.

Dhaene, J. (1989). Stochastic Interest Rates And Autoregressive Integrated Moving Average Processes. *ASTIN BULLETIN*, 19(2): 131-138.

Dufresne, D. (1990). The Distribution of a Perpetuity with Applications to Risk Theory and Pension Funding, *Scandinavian Actuarial Journal*, 9, 39-79.

Giacotto, C. (1986). Stochastic modelling of interest rates: Actuarial vs. equilibrium approach. *Journal of Risk and Insurance*, 53, 435-453.

Lin, L. (2008) Multivariate Stochastic Analysis of a Combination Hybrid Pension Plan. *MSc. Project, SFU*

Pandit, S.M. and Wu, S. (1983). *Time Series and System Analysis with Applications.* John

Wiley & Sons, New York.

Panjer, H.H. and Bellhouse, D.R. (1980). Stochastic Modelling of Interest Rates With Applications To Life Contingencies . *Journal of Risk and Insurance*, 47: 91-110.

Parker, G. (1993). Distribution of the present value of future cash flows. *Proceedings of the 3rd AFIR International Colloquium*, Rome, 831-843.

Parker, G. (1994a). Two Stochastic Approaches For Discounting Actuarial Functions. *ASTIN BULLETIN*, 24(2): 167-181.

Parker,G. (1994b). Limiting distribution of the present value of a portfolio. *ASTIN Bulletin* 94(1), 47-60.

Parker, G. (1995). A Second Order Stochastic Differential Equation For The Force of Interest. *Insurance: Mathematics and Economics*, 16: 211-224.

Pollard, J.H. (1971). On fluctuating interest rates. *Bulletin de L'Association Royale des Actuaires Belges 66*, 68-97

Vasicek, O.A. (1974) An Equilibrium Characterization of the Term Structure. *Journal of Financial Economics*, 5, 177-188

Wan, Z. (2010). Modeling Investment Returns With A Multivariate Ornstein-Uhlenbeck Process. *MSc. Project, SFU*

Waters, H.R. (1978) The Moments and Distributions of Actuarial Functions. *Journal of the Institute of Actuaries* 105, Part I, 61-75.