# MULTIVARIATE STOCHASTIC ANALYSIS OF A COMBINATION HYBRID PENSION PLAN


by

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# Abstract

A combination hybrid pension plan that consists of a defined contribution account and a final salary defined benefit guarantee is studied by using multivariate time series analysis. This time series include salary increase, inflation rate and investment return. The loss function for the plan sponsor is defined, its first three conditional moments are derived and its distribution is approximated. Different investment strategies for the DC account are compared. A simulation study is also performed for illustration and validation purposes. Finally, the concept of Economic Capital is introduced to perform risk management on this pension plan.

**Keywords:** Hybrid Pension Plan, Multivariate Time Series Analysis, Gaussian Process, Limiting Portfolio, Simulation, Economic Capital.

# Dedication

To my parents.

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# Chapter 1

# Introduction

## 1.1 Hybrid Pension Plans

As the main source of income for most retirees, pension plans play an important role in our lives in terms of maintaining our lifestyle after retirement. Therefore when it comes to pension plans, how to identify and measure the underlying risks are essential for both the plan sponsor and its plan members. Pension plans can be categorized as Defined Benefit (DB), Defined Contribution (DC) and hybrid plans based on their specific plan design. In a DB plan, the benefit payment after one's retirement is stated by the policy at issue and the contribution rate is then evaluated accordingly on regular basis as required by local regulations. The situation is reversed in a DC plan: the contribution rate, which is usually a percentage of pre-tax salary, is set to a constant at issue and all the contributions made for one person will be put into a DC account. The accumulated account balance at one's retirement will determine his or her benefit payment after retirement. Hybrid pension plan designs are those that are neither a full DB nor a full DC plan. For example, there could be a DC account for each individual plan member to invest the contributions, but the benefit payment could take form of a DB plan.

One appealing feature of DB plan designs for employees is that it protects its plan members against risks associated with the investment returns earned by the contributions. Such risks are designed to fall on the plan sponsor, in most cases, the employers, since they are responsible for the benefit payment even if the financial market performs badly or insufficient contributions were made. On the other hand, DC plan designs are favored by employers in the sense that they don't need to worry about the benefit payments at all, while the plan members are exposed to almost all possible risks that are related to the DC account or the pension payment. This is probably the main reason why the implementation of DC plan designs has always been controversial. As for hybrid plans, the risk allocation between plan sponsor and plan member varies with each design. Take a Cash Balance plan which is currently the most popular hybrid plan in the US for example, the plan sponsor undertakes the investment risk usually through the guarantee of an investment return on the pensioner's DC account, while the plan members undertake the annuity conversion risk and salary inflation risk since they will be given a lump-sum instead of a life annuity upon retirement, based on their contribution history which is closely related to salary inflation.

Pension plans in the early days were mostly DB plans, and most government level pension plans nowadays are DB plans. Social Security in the US and Canada Pension Plan, Old Age Security in Canada are all defined benefit plans. However, for employer-sponsored plans, there has been a notable shift from DB plans to hybrid and DC plans for the past few decades. Hewitt Bacon and Woodrow (2005) explained that the reason for the decline of final salary scheme (one of the most important DB schemes) in UK is the fact that the plan sponsors want to reduce the volatility of costs for a pension plan, which leads to the decision of reducing costs. Those costs were originally increased by low interest rates and investment returns, improved longevity, and improvements in pension benefits. MacDonald and Cairns (2006) concluded that the shift toward DC design is mainly due to the simplicity and portability of a DC design, the risk reduction to plan sponsors, and the opportunity of less contribution as well as to avoid the rising cost of DB designs. They also pointed out that the main drawbacks of a pure DC plan is the uncertainty in the level of pension benefit due to fluctuations of the investment return in the DC account. This is also one of the reason why the shift from DB to hybrid plans is better accepted. The risk sharing between plan sponsor and plan member of hybrid pension is more even than both DB and DC plans.

Three ”common” hybrid designs are:

- **Cash Balance**: The plan member is entitled to a capital sum at retirement and the lump sum is converted to life annuity just like in DC plans. However the balance in each member’s account is not directly decided by the underlying asset, it could be a guaranteed value or subject to certain form of underwriting by the plan sponsor. In this design, the plan member is exposed to annuity conversion risks the same way as DC plan, but is protected against some of the investment risks.

- **Career Average Plans:** Also known as Index Pension Plans. The pension benefit is revalued every year based on the average salary throughout the plan member's career. In this plan design, risks associated with inflation and real wage increase is reduced for the plan member since the pension benefit depends on the actual final salary.

- **Combination Hybrids:** The pension benefit for such products can accrue on two basis. Here we assume a DC basis and a final salary basis. The plan member will choose one of the pension benefits upon retirement. The final salary is also subject to reevaluations in this study. Therefore the inflation risks, real wage increase risks, annuity conversion risks as well as investment risks are all shared between plan sponsor and its members. We can also consider this combination hybrid as a DC plan with a final salary scheme guarantee. This is the hybrid product we will investigate into details.

We will build up time series models for different risk factors in a combination hybrids plan to model the loss function of the plan sponsor on each individual policy to assess the embedded risks for such a final salary guarantee.

## 1.2 Actuarial Applications of Interest Rate Models

In this study we applied stochastic analysis of multivariate time series variables to actuarial functions of a combination hybrids product. The inflation rate, real wage increase, investment return and long term treasury bond return are modeled in their continuously compounded forms to "discount" future cash flows to the time of issue.

Many one dimensional time series models have been applied to evaluate actuarial functions. One of the earliest models applied for the force of interest is the White Noise process which assumes that the force of interest for different time periods are identically and independent distributed from a normal distribution. Waters (1978) worked under such framework to model cash flows with life contingencies and obtained the first four moments of some actuarial functions. The Pearson curve was also employed to fit limiting distributions of those actuarial functions.

Besides independent time series models, autoregressive processes have also been discussed to model the force of interest in an actuarial context. Panjer and Bellhouse (1980) considered both discrete autoregressive models and their continuous equivalent models, stochastic differential equations, and showed how to obtain unconditional moments of actuarial functions under these models where the orders of those process include one and two. The results were extended by taking historical data into account and applying conditional probability measurements in Bellhouse and Panjer (1981). A more general discrete time series model–Autoregressive Integrated Moving Average process (ARIMA process) was introduced by Dhaene (1989) to model the force of interest.

Besides the force of interest, some previous studies preferred to model the force of interest accumulation function with a time series process. Beekman and Fuelling (1990) used an Ornstein-Uhlenbeck process to model the force of interest accumulation function and derived the first two moments of both deterministic and contingent future cash flows.

Parker(1994b) investigated both modeling approaches, to model the force of interest or to model the force of interest accumulation function, with White Noise process, Wiener process and Ornstein-Uhlenbeck process. Formula and numerical illustrations were presented to show that those two approaches were by no means equivalent. When it comes to conditional probability measurements, modeling the force of interest could take current market conditions into account while modeling the force of interest accumulation function would simply ignore those information .

Parker(1993a, 1993b, 1994a, 1996, 1997) introduced derivation for moments of present value variables and a non-parametric method to obtain approximated distribution function of annuity certain and limiting portfolio of insurance policies including endowment, temporary and whole life contracts. This approximation method is extended to two dimensional time series variables in this study to obtain the cumulative density function of the loss variable of combination hybrid plans.

Since a pension plan usually involves risk factors other than the force of interest and mortality, such as inflation rate and real wage increase, there have been many attempts to apply multivariate time series processes in the valuation of pension plans. MacDonald and Cairns (2006) presented a simulation study based on multivariate models of continuously compounded rate of return, the CPI log growth and the real log return on wages to investigate the impact of nationwide implementation of pure DC pension scheme on the population dynamic. By making many ideal assumptions, the authors conclude that the nationwide implementation of a pure DC plan design would cause significant volatilities in the population's retirement dynamic when early retirement is allowed. For example when the financial market is offering high returns to the DC account, lots of people would choose to retire earlier than the regular retirement age 65.

Sherris (1995) constructed a multivariate model to evaluate option features in retirement benefits under the no-arbitrage assumption. The author argued that the lattice model which was used in contingent claims valuation of financial options were not computationally feasible for retirement benefits, while a crude simulation which was considered to be more efficient than the lattice model showed that traditional deterministic valuation understated the cost of providing those guarantees in pension plans by as much as 35 percent.

In this study, we will first construct a four dimensional vector autoregressive process of order one to the main risk factors involved in a combination hybrid plan, and then obtain conditional moments and an approximation of the distribution of the loss function through some linear transformations of modeled variables.

## 1.3 Outline

In Chapter Two, we will introduce the detailed features of a combination hybrid pension plan with a final salary scheme. The assumption of a limiting portfolio is made to get cash flows for averaged individual policy. Illustrations of future cash flows and variables that need to be modeled are provided.

Then we will define a multivariate time series model VAR(1) in Chapter Three and derive conditional moments for the four-dimensional variable. Three investment strategies are proposed for the DC account and three VAR(1) models corresponding to those strategies are constructed based on historical data from the US financial market to use for our illustrations in later chapters.

We combine the loss function of the combination hybrid pension plan and the VAR(1) model in Chapter Four to investigate the randomness of the potential loss function from the plan sponsor's perspective. Through linear transformations and summations of the modeled variables, the first three conditional moments of the loss function are obtained and numerical results are shown.

The distribution of the average loss function is then studied in Chapter Five. The approximation method proposed by Parker(1993a) is extended to multivariate cases and illustrations are provided with a short term pension plan. Results are then checked with the theoretical moments and simulations. The simulation method is also applied to get the distribution of the loss function. Comparisons between these two methods are then made.

In Chapter Six, we introduce the concept of risk measure and Value-at-Risk based Economic Capital to provide some perspectives to plan sponsors regarding the amount of risk capital that needs to be set aside to maintain an acceptable probability of staying solvent. Different investment strategies are compared to show the impact of asset allocation in the DC account.

Finally, the main conclusions from this study are discussed in Chapter Seven.

# Chapter 2

# Combination Hybrid Pension Plan

Now we introduce a combination hybrid plan which has a DC account and a pension income guarantee through a minimum replacement ratio. Contributions are made at the beginning of each year as a percentage of pre-tax annual salary and pension benefit payments start on normal retirement date and also paid at the beginning of year.

## 2.1 Basic Plan Design

The replacement ratio $RR$ for a pension plan is defined as

$$RR = \frac{\text{Annual Pension Benefit}}{\text{Annual Final Salary before Retirement}} \tag{2.1}$$

We assume that this combination hybrid plan offers a minimum guarantee on the replacement ratio rather than the amount of pension benefit. We believe that the replacement ratio is a more accurate measurement of the quality of life that the pension offers than a deterministic pension benefit amount since it is relative to the final year salary. The main function of a pension plan should be to enable its members to maintain more or less the same lifestyle in retirement, and the quality of the lifestyle could be expressed in terms of their final salary. Under such a plan design, the risks associated with inflation and salary increase are shared by the plan sponsor and its members. For example, when the real wage increase is quite low compared to the contemporary inflation rate during one's career, the guaranteed pension benefit would not provide a sufficient income after retirement. In this case, the plan member could only hope that the investment in the DC account would provide relative high returns. MacDonald and Cairns (2006) discussed that a replacement ratio between 60% and 74% would be sufficient for retirees to maintain their pre-retirement standard of living, because there are some work associated cost that can be reduced after retirement. In this study, we will apply a 70% replacement ratio guarantee for illustration purposes.

Withdrawals from this hybrid plan before retirement or late retirements will result in lump-sum payments of the DC account at that time, in such cases the sponsor acts more like a fund manager who does not carry any risk. Therefore from the sponsors' point of view, the only situation we need to take into account is when the plan member enters the pension plan at the start of his career and retires at normal retirement age. We also make the assumption that there are no transaction fees, expenses, commissions or taxes for simplicity.

## 2.2 Plan Feature Assumptions

Now let us take a single policy of this combination hybrid plan for example to investigate in details. Assume that John Doe starts his career at age 25 and enters this pension plan immediately. He will stay in this plan until normal retirement age of 65 and start receiving pension benefits from age 65.

### 2.2.1 Contributions

Annual contributions are made to the DC private account as a percentage, $100c\%$, of John Doe's annual salary. This contribution is usually a sum of amounts from both the plan sponsor and John Doe himself. MacDonald and Cairns (2006) stated that previous research in the United States showed that a contribution rate in the range $8.7\% \leq c \leq 12.6\%$ is acceptable. In this study, $c$ is initially set to 10% for illustration purposes.

There is one specified investment strategy assigned to each individual DC account and this strategy is kept the same for all contributions throughout his career. We will consider 1-year treasury bonds and stocks as the only two choices of asset. By changing the weights of these two assets in the DC account we can have different asset allocations. Here we assume that the asset allocation in the DC account remains the same throughout the contribution phase. Therefore rebalancing will be done at year end, right before the next contribution, to make sure that the target asset allocation is maintained.

### 2.2.2 Guaranteed Pension Benefit

Given the replacement ratio guarantee, upon retirement John Doe will get the maximum of the guaranteed benefit payment which is the product of the guaranteed replacement ratio and his final salary, and the pension benefit he can purchase in the life annuity market with his DC account balance at that time as his pension benefit. No further indexing is applied to the pension benefit after retirement.

No attempts have been made in this study to model mortality risks in this product. Therefore one more assumption made for simplicity is that this plan has a huge number of participants in each age group. Therefore the number of deaths in each age group exactly follows the mortality rate in the life table. Such a portfolio of policies is called a limiting portfolio. Under this assumption, the plan sponsor will be facing a set of deterministic pension payments in John Doe's policy once he reaches age 65.

## 2.3 Cash Flows Illustration

Now let us take a look at future cash flows in this pension from both John Doe's and the plan sponsor's perspectives to get a better understanding.

Let $S(t)$ denote John Doe's salary at age $t$, $F(t)$ denote the balance of his DC account at age $t$, and $a$ denote the market price of a life annuity at his retirement. Figure 2.1 gives an illustration of future cash flows that John Doe would expect. Contributions of 10% of John Doe's annual salary are made to this DC account until age 65. Upon retirement, he will either choose the guaranteed benefit $70\%S(65)$ or purchase a life annuity with annual benefit $F(65)/a$. There are in total 40 annual contributions while the number of pension payment is a random integer that depends on the age of death for John Doe.

Figure 2.2 presents the cash flows for each individual policy from the plan sponsor's perspective. Cash flows of the annual contribution is the same as John Doe. The

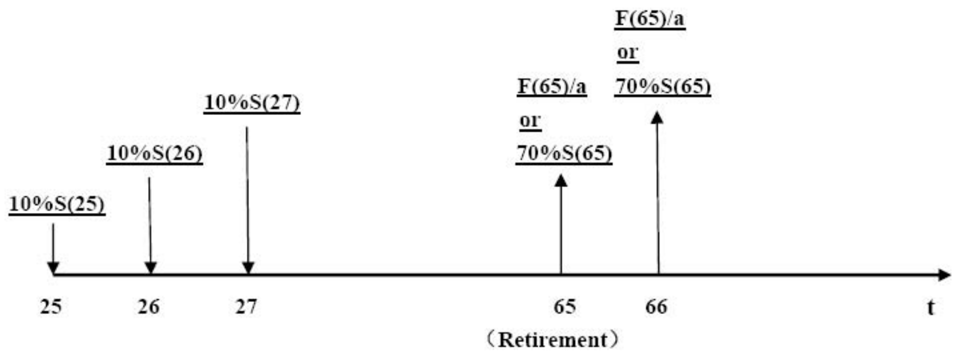


Figure 2.1: Cash flows in the plan studied from John Doe's perspective

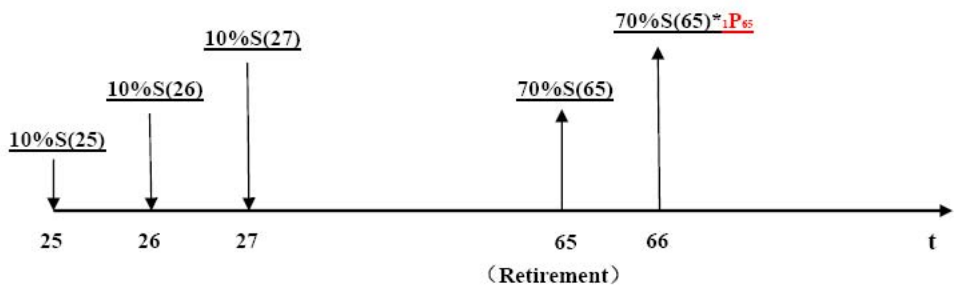


Figure 2.2: Cash flows in the plan studied from the plan sponsor's perspective

situation is slightly different regarding the pension benefit payments. While providing a salary related benefit guarantee, the plan sponsor only needs to consider the cases where the guarantee is exercised and evaluate the loss function of each policy accordingly. Further, since the sponsor has a limiting portfolio, the averaged guaranteed annuity payment in each single policy should be multiplied by a survival probability ${}_{t-65}p_{65}$ at age $t$ of John Doe, and the averaged future cash flows will terminate at the end of the life table when the survival probability hits zero. Take the 2001 CSO table (see Appendix A) for example, since $q_x = 1$ for $x = 120$, there will be $120 - 64 = 56$ positive pension payments in each individual policy.

## 2.4 Risk Allocations

With the combination of final salary scheme and DC scheme, this plan provides a wide range of risk sharing between the plan sponsor and its members. Risks associated with inflation, real wage increase and investment in the DC account are all shared at different levels.

For John Doe, there is not much explicit risk related to inflation before his retirement or investment returns in the DC account as long as he is satisfied with the guaranteed replacement ratio. However he is exposed to risks associated with salary increase and inflation after retirement. If the salary increases do not follow contemporary inflation rates and the investment in the DC account performs badly which results in the exercise of the guarantee, John Doe might suffer a quite poor retirement just as his poor pre-retirement life standard. Another undesirable situation is that the inflation rate shoots up after his retirement but his pension payments are not further indexed, life will become tougher as he ages.

According to the combination hybrid design, the plan sponsor certainly takes on much more risks than a simple DC plan. Exposures to inflation risks, investment risks and real wage increase risks are all brought in by the final salary guarantee. Actually this design is quite close to a DB final salary scheme in which the plan sponsor needs to prepare against all the risks mentioned above and other risk factors. The main appeal for the plan sponsor of this combination hybrids pension is that when the investment return in the DC account is higher than the salary increase plus inflation, there is a great chance that the guarantee won't be exercised and it is almost like in a DC plan design. This study will focus on the sponsor's obligations with respect to this combination hybrid plan. The inflation rate, investment return, salary increase and long term treasury bond return will be modeled by a multivariate time series model. No attempts have been made in this study to model mortality risks such as catastrophic events and longevity risk. We assume a limiting portfolio so that the study can show the impact of the stochastic economic variables mentioned above.

# Chapter 3

# Multivariate Time Series

The four variables stated in the last chapter are modeled with a four dimensional multivariate time series in this study. There are many stochastic processes, both discrete and continuous, in the literature that are employed to model variables such as salary increase, inflation rate, and investment returns. A vector AR(1) model is chosen here to model the economic variables that are involved in this hybrid pension plan and to investigate the stochastic loss function at issue for the plan sponsor. The vector AR(1) model is actually equivalent to a continuous multivariate Ornstein-Uhlenbeck process which is a stochastic differential equation of order one. When the dimension of the variable is one, the Ornstein-Uhlenbeck process is also known as the Vasicek model in finance.

First let us take a brief look at the so called Vasicek model and AR(1) model that are commonly used in modeling one-dimensional variable, such as the rate of return or the interest rate, in economic, finance and actuarial studies.

- Vasicek Model

  The random variable $r_t$ which represents the short rate at time $t$ is said to follow the Vasicek Model if the following stochastic differential equation holds:

$$dr_t = a(b - r_t)dt + \sigma dW_t \tag{3.1}$$

  where $W_t$ is a Wiener process that models the market risk, $\sigma$ is the instantaneous volatility, $b$ is the long-term mean of this process and $a$ represents the mean-reversion coefficient.

- AR(1) Process

  The short rate $r_t$ is said to follow an AR(1) process if

$$r_t - \mu = \phi(r_{t-1} - \mu) + \epsilon_t \tag{3.2}$$

  where $\epsilon_t$ is the white noise term, $\mu$ is the long term mean of this AR(1) process.

  The AR(1) process is a simple and commonly used member of the ARMA(p,q) family. The process $\{r_t\}$ is stationary if and only if $|\phi| < 1$.

Even though the Vasicek model is a continuous differential equation while the AR(1) process is defined in discrete time, it has been proved they are actually equivalent processes according to the principle of covariance equivalence. Any AR(1) process with $0 \leq \phi \leq 1$ can be viewed as the discrete representation of a Vasicek model while any Vasicek model has a discrete analogue model that is an AR(1) process. In model estimation, most data available are actually discrete. Therefore the equivalence relation between a Vasicek model and an AR(1) process enables the estimate of continuous Vasicek model with discrete observations of $r_t$.

In this study, a four dimensional vector AR(1) discrete process is employed instead of the Vasicek model since the data we have are all discrete and our study for the loss function will be based on discrete time.

## 3.1 Variables In Scope

Now let us take a detailed look at the time series variables involved in this study:

- $\alpha_i$ : the inflation free continuously compounded salary increase from time $i$ to $i+1$
- $f_i$ : the continuously compounded inflation rate from time $i$ to $i+1$
- $\delta_i$ : the inflation free continuously compounded investment return from time $i$ to $i+1$
- $l_i$ : the inflation free continuously compounded rate of return for 10-year treasury bond at time $i$.

The vector $\underline{X_t} = (\alpha_t, f_t, \delta_t, l_t)$ is modeled by a four dimensional Vector AR(1) model and estimates are obtained based on data from the US financial market. Detailed estimates will be presented in later sections.

Now we will introduce how the variable $\underline{X_t}$ is involved in this combination hybrid plan. According to the previous chapters, the plan sponsor needs to consider the following average cash flows in each individual policy:

- **Contributions**:

  Every year throughout John Doe's career, for example at age $t$, where $25 \leq t \leq 64$, there will be a contribution of $10\% \cdot S(t)$ made to his DC account, where $S(t) = S(25) \cdot e^{\sum_{i=25}^{t-1} \alpha_i + f_i}$. The initial salary, $S(25)$, is a constant known at plan issue date.

- **Investment return in the DC account**:

  The annual investment return in this DC account from age $t$ to $t+1$ is $\delta_t + f_t$. As discussed earlier, annual rebalancing is performed which assures that this return, $\delta_t + f_t$, is realized every year.

- **Pension benefit payable**:

  Upon retirement, the guaranteed pension income will be defined as

  $RR \cdot S(65) = RR \cdot S(25) \cdot e^{\sum_{t=25}^{64} \alpha_t + f_t}$

  Since we assume that salary payments occur at beginning of each year, $S(65)$ is not really a salary payment since John Doe should have retired by then, but we still use the projected $S(65)$ to define his pension benefit to include the market information from age 64 to 65.

- **Retirement decisions**:

  The pensioner will choose between the guaranteed benefit and purchasing a life annuity with a lump-sum from the DC account at retirement. The market price for a unit life annuity at that time is $\ddot{a}_{65} = 1 + \sum_{k=1}^{w-65} {}_kp_{65} \cdot e^{\sum_{i=65}^{k+64} -(l_i + f_i)}$ which leads to an annual benefit of $F(65)/\ddot{a}_{65}$.

## 3.2 Vector AR(1) Models

Now we will introduce the Vector AR(1) model. A vector autoregressive time series of order one is defined as follows:

**Definition 3.1.** *An n-dimensional time series random variable* $\underline{X_t}$ *is said to follow an n-dimensional vector autoregressive model of order one, VAR(1) model, if*

$$\underline{X_t} - \underline{\mu} = \underline{\Phi} \cdot (\underline{X_{t-1}} - \underline{\mu}) + \underline{a_t}, \tag{3.3}$$

*where* $t \in \mathbb{Z}$, $\underline{\mu}$ *is the long-term mean vector of* $\underline{X_t}$, $\underline{\Phi}$ *is the autoregressive coefficient matrix and* $\underline{a_t}$ *is the white noise term which means that* $\{\underline{a_t} | t \in \mathbb{Z}\}$ *are identically and independently distributed and follows a n-dimensional multivariate normal distribution with zero mean and covariance matrix* $\underline{\Sigma}$.

The stationary property of a time series model is essential when it comes to applications in a stable economic environment. A sufficient condition of a stationary VAR(1) process is given as follows (see section 11.3 of Brockwell and Davis (1991)):

**Theorem 3.1.** *A VAR(1) model is stationary if all the eigenvalues of* $\underline{\Phi}$ *are less than 1 in absolute value, i.e. provided det*$(I - z\underline{\Phi}) \neq 0$ *for all* $z \in \mathbb{C}$ *such that* $|z| \leq 1$.

The unconditional moments of a Vector AR(1) process have been discussed in many textbooks. Since we will construct the model out of historical data, it seems more reasonable to use conditional probability measurements for practical purposes. The following theorem gives the results for the first two conditional moments of a stationary VAR(1) model.

**Theorem 3.2.** *For an n-dimensional VAR(1) model satisfying Equation (3.3), the first two conditional moments of $\underline{X_t}$ given $\underline{X_0}$ can be obtained as follows:*

$$\begin{aligned} E(\underline{X_t}|\underline{X_0}) &= \underline{\Phi}^t(\underline{X_0} - \underline{\mu}) + \underline{\mu} \\ Cov(\underline{X_t}, \underline{X_{t-k}}^T|\underline{X_0}) &= \sum_{i=1}^{t-k} \underline{\Phi}^{t-i} \cdot \underline{\Sigma} \cdot (\underline{\Phi}^{t-k-i})^T. \end{aligned} \tag{3.4}$$

**Proof:** For the first conditional moment, if $t = 1$, then

$$E(\underline{X_1}|\underline{X_0}) = E(\underline{\Phi} \cdot (\underline{X_0} - \underline{\mu}) + \underline{\mu} + \underline{a_1}|\underline{X_0}).$$

Since $\underline{a_1}$ has mean zero, we have

$$\begin{aligned} E(\underline{X_1}|\underline{X_0}) &= E(\underline{\Phi} \cdot (\underline{X_0} - \underline{\mu})|\underline{X_0}) + \underline{\mu} \\ &= \underline{\Phi} \cdot (\underline{X_0} - \underline{\mu}) + \underline{\mu}. \end{aligned}$$

When $t > 1$,

$$\begin{aligned} E(\underline{X_t}|\underline{X_0}) &= E(\underline{\Phi} \cdot (\underline{X_{t-1}} - \underline{\mu}) + \underline{\mu} + \underline{a_t}|\underline{X_0}) \\ &= E(\underline{\Phi} \cdot (\underline{X_{t-1}} - \underline{\mu})|\underline{X_0}) + \underline{\mu} \\ &= E(\underline{\Phi}^2 \cdot (\underline{X_{t-2}} - \underline{\mu}) + \underline{\Phi} \cdot \underline{a_{t-1}}|\underline{X_0}) + \underline{\mu} \\ &= E(\underline{\Phi}^2 \cdot (\underline{X_{t-2}} - \underline{\mu})|\underline{X_0}) + \underline{\mu} \\ &\quad \cdots\cdots \\ &= E(\underline{\Phi}^t \cdot (\underline{X_0} - \underline{\mu})|\underline{X_0}) + \underline{\mu} \\ &= \underline{\Phi}^t \cdot (\underline{X_0} - \underline{\mu}) + \underline{\mu}. \end{aligned}$$

As for the second conditional moment, since $\underline{X_t}$ can be written as

$$\underline{X_t} = \underline{\Phi}^t \cdot (\underline{X_0} - \underline{\mu}) + \sum_{i=1}^{t} \underline{\Phi}^{t-i}\underline{a_i},$$

for any $k \in \mathrm{N}$, we have

$$\begin{aligned} E(\underline{X_t}\underline{X_{t-k}}'|\underline{X_0}) &= E\Bigg[\Bigg(\underline{\Phi}^t \cdot (\underline{X_0} - \underline{\mu}) + \underline{\mu} + \sum_{i=1}^{t} \Phi^{t-i}\underline{a_i}\Bigg)\Bigg(\underline{\Phi}^{t-k} \cdot (\underline{X_0} - \underline{\mu}) \\ &\quad + \underline{\mu} + \sum_{j=1}^{t-k} \Phi^{t-k-j}\underline{a_j}\Bigg)^T \Bigg|\underline{X_0}\Bigg] \\ &= \Big(\underline{\Phi}^t(\underline{X_0} - \underline{\mu}) + \underline{\mu}\Big) \cdot \Big(\underline{\Phi}^{t-k}(\underline{X_0} - \underline{\mu}) + \underline{\mu}\Big)^T \\ &\quad + E\Bigg[\Bigg(\sum_{i=1}^{t} \underline{\Phi}^{t-i}\underline{a_i}\Bigg) \cdot \Bigg(\sum_{j=1}^{t-k} \underline{\Phi}^{t-k-j}\underline{a_j}\Bigg)^T \Bigg|\underline{X_0}\Bigg]. \end{aligned}$$

Since $\underline{a_t}'s$ are i.i.d.

$$\begin{aligned} E(\underline{X_t}\underline{X_{t-k}}^T|\underline{X_0}) &= \Big(\underline{\Phi}^t(\underline{X_0} - \underline{\mu}) + \underline{\mu}\Big) \cdot \Big(\underline{\Phi}^{t-k}(\underline{X_0} - \underline{\mu}) + \underline{\mu}\Big)^T \\ &\quad + \sum_{i=1}^{t-k} \Bigg[\underline{\Phi}^{t-i} \cdot \underline{\Sigma} \cdot \Big(\underline{\Phi}^{t-k-i}\Big)^T\Bigg]. \end{aligned}$$

Therefore,

$$\begin{aligned} Cov(\underline{X_t}, \underline{X_{t-k}}^T|\underline{X_0}) &= E(\underline{X_t}\underline{X_{t-k}}'|\underline{X_0}) - E(\underline{X_t}|\underline{X_0})E(\underline{X_{t-k}}|\underline{X_0})^T \\ &= \sum_{i=1}^{t-k} \Bigg[\underline{\Phi}^{t-i} \cdot \underline{\Sigma} \cdot \Big(\underline{\Phi}^{t-k-i}\Big)^T\Bigg]. \qquad \square \end{aligned}$$

Since this is a Guassian process, we can obtain the conditional distribution $f_{\underline{X_t}|\underline{X_0}}(\underline{x_t}|\underline{x_0})$ for $\underline{X_t}$ only through the first two moments. Therefore once we have these two moments, the basic modeling for $\underline{X_t}$ given $\underline{X_0}$ is completed. Next we will estimate the VAR(1) model with data from the US market.

## 3.3 Model Estimation

Having chosen the time series process for modeling, we now move on to construct models with historical data from the US financial market.

### 3.3.1 Data Collection

In order to derive a practical model, we choose to use real world data to fit VAR(1) models. All data are collected quarterly for the past 20 years to build a discrete model where one unit of time stands for a quarter. Annual inflation rates, real wage increases and 10-year treasury bond returns are converted to continuously compounded rates before the estimation.

As for the investment return of the DC account, we make some simple assumptions about the asset available for investment. Only two assets are selected as general representatives, the S&P 500 Index and 1-year treasury bond. The S&P 500 Index is chosen to represent asset with high average return and high volatility while 1-year treasury bond stands for assets with low average return and low volatility. Different investment strategies in the DC account are realized by setting different weights for those two assets. After calculating the weighted annual return, the equivalent continuously compounded rate is obtained.

Table 3.1 shows the asset allocations that are studied here. Strategy A has the lowest weight in stocks which makes it a low risk portfolio while strategy C represents a high risk portfolio and strategy B represents a medium risk portfolio.

| | Strategy A | Strategy B | Strategy C |
|---|---|---|---|
| S&P 500 | 0.2 | 0.5 | 0.8 |
| 1-year treasury bond | 0.8 | 0.5 | 0.2 |

Table 3.1: Asset Allocations in the DC account

Even though data for inflation rates, real wage increases and 10-year treasury bond returns are the same for all three investment strategies, we choose to model those strategies separately with three VAR(1) models since the correlation matrix can be very different. Therefore three set of parameters will be obtained for the VAR(1) process based on the data obtained for each investment strategy.

### 3.3.2 Estimation Method

Following are a few key steps we take to obtain parameters for the VAR(1) model:

1. **Long Term Mean**: The sample mean $\underline{\bar{x}}$ is used as an estimate for the long term mean $\underline{\mu}$ of this process.

2. **Estimates for $\underline{\Phi}$**: Estimation of the matrix $\underline{\Phi}$ is done with the default Yule-Walker method in R programme after subtracting the sample mean. Then the condition in Theorem 3.1 will be examined to make sure that the process is stationary.

3. **Estimates for $\underline{\Sigma}$:** As for the covariance matrix for $\underline{a_t}$, the covariance matrix of the residuals from step 2 is calculated and used as an estimate for $\underline{\Sigma}$.

4. **Initial Value:** The latest observation for $\underline{X_t}$ is used as the initial value $\underline{X}^0$ and it represents the value for $\underline{X_t}$ at issue.

### 3.3.3 Estimation Results

First let us take a look at the quarterly historical data that has been collected from the last twenty years. (See Appendix B)

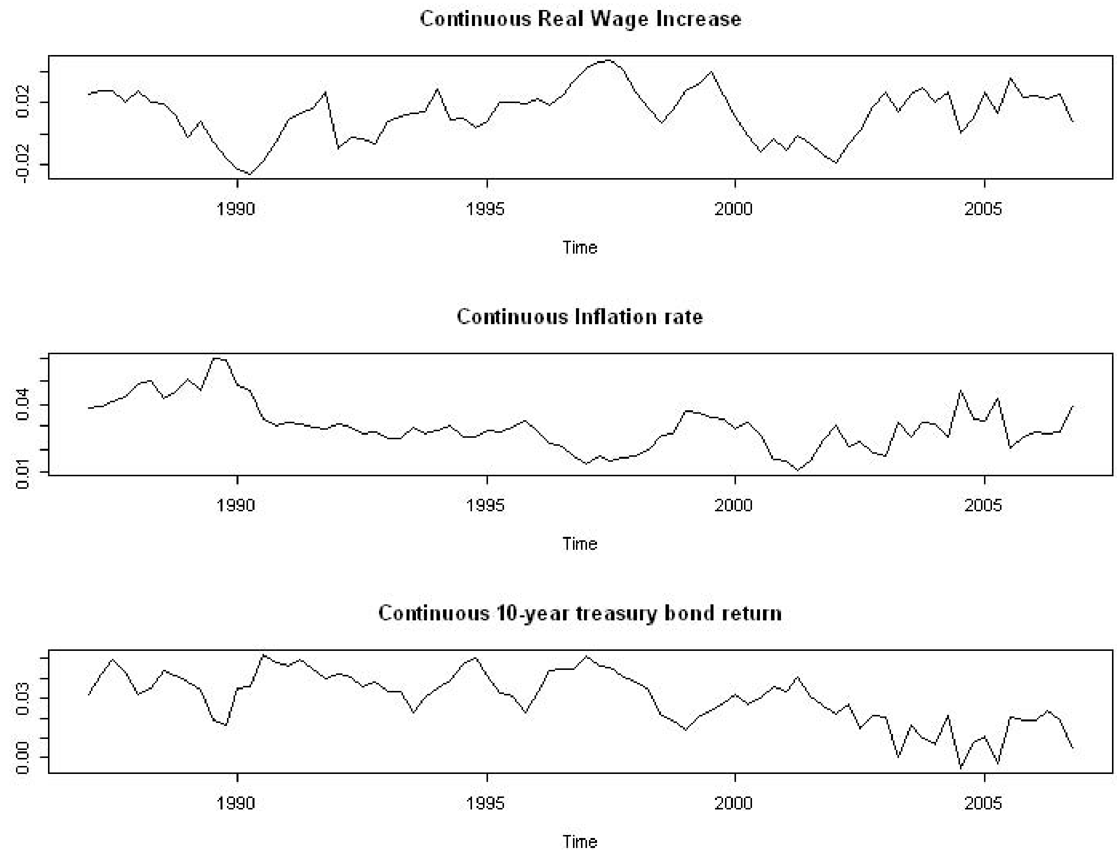


Figure 3.1: Quarterly historical data for salary increase, inflation and long-term bonds

Figure 3.1 shows the time series plot for continuous compounded real wage increase, inflation rate and real long-term treasury bond returns. We can find some similarity in the shape between real wage increase and real long-term treasury bond returns through time, but inflation rate seems to move in opposite directions with the other two.

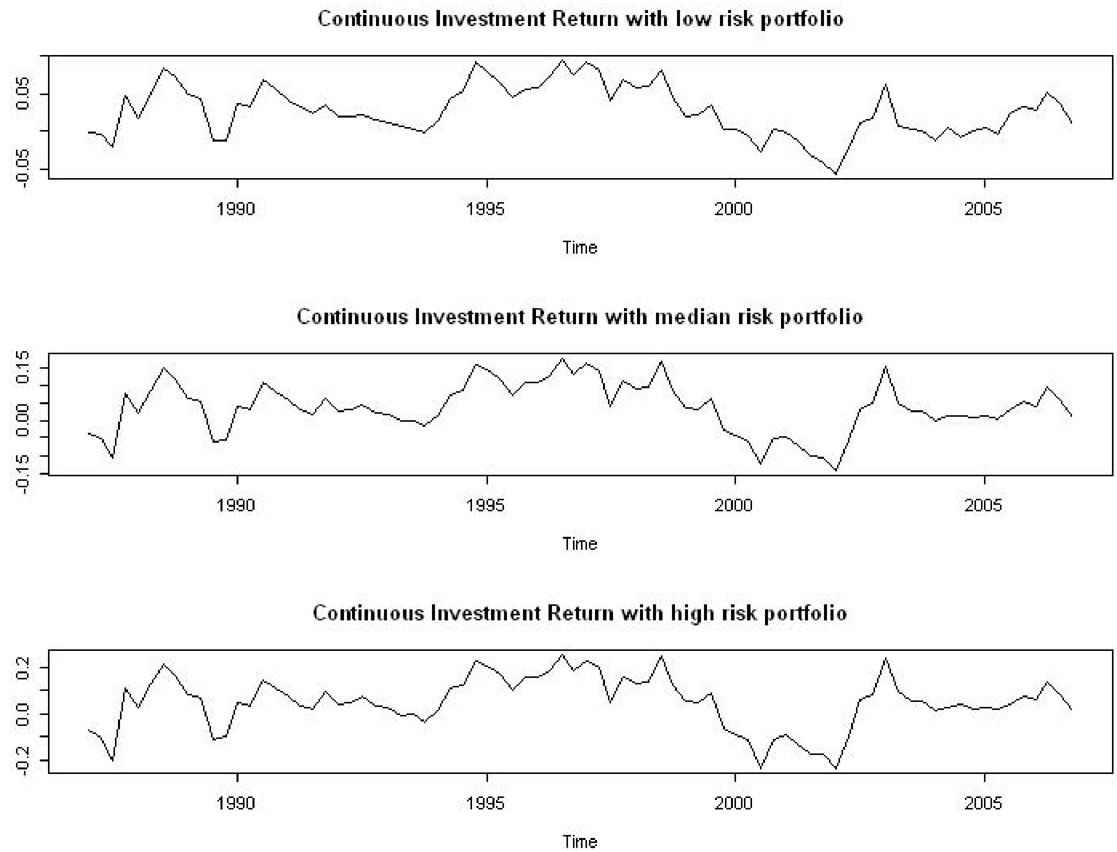


Figure 3.2: Quarterly historical data for different investment strategies

Data for different investment strategies in the DC account are shown in Figure 3.2. The trend behavior of returns for different investment strategies does not vary a lot through the time horizon we chose, but there are some slight differences on the volatility among them. We will find further validations about the volatility in the model estimates.

Now let us have a look at the estimates from those data. The quarterly VAR(1) models obtained are as follows:

- Low risk investment strategy A:

$$\underline{X_t} - \underline{\mu} = \begin{pmatrix} 0.77 & 0.074 & 0.066 & -0.057 \\ 0.11 & 0.87 & -0.011 & 0.089 \\ 0.044 & 0.14 & 0.74 & 0.096 \\ -0.042 & 0.21 & -0.00052 & 0.80 \end{pmatrix} \cdot \left(\underline{X_{t-1}} - \underline{\mu}\right) + \underline{a_t},$$

where $\underline{X}^0 = (0.0072, 0.040, 0.0098, 0.0046)^T$, $\underline{\mu} = (0.013, 0.030, 0.027, 0.030)^T$ and the covariance matrix for $\underline{a_t}$ is

$$\underline{\Sigma} = \begin{pmatrix} 10.63 & -1.43 & 4.49 & 3.77 \\ -1.43 & 5.19 & -6.08 & -2.26 \\ 4.49 & -6.08 & 43.36 & 6.76 \\ 3.77 & -2.26 & 6.76 & 7.36 \end{pmatrix} \cdot 10^{-5}.$$

- Medium risk investment strategy B:

$$\underline{X_t} - \underline{\mu} = \begin{pmatrix} 0.76 & 0.072 & 0.036 & -0.035 \\ 0.11 & 0.87 & -0.0029 & 0.081 \\ 0.12 & 0.093 & 0.71 & 0.26 \\ -0.038 & 0.21 & -0.0029 & 0.80 \end{pmatrix} \cdot (\underline{X_{t-1}} - \underline{\mu}) + \underline{a_t},$$

where $\underline{X}^0 = (0.0072, 0.040, 0.012, 0.0046)^T$, $\underline{\mu} = (0.013, 0.030, 0.038, 0.030)^T$ and the covariance matrix for $\underline{a_t}$ is

$$\underline{\Sigma} = \begin{pmatrix} 10.45 & -1.44 & 7.94 & 3.80 \\ -1.44 & 5.19 & -11.11 & -2.26 \\ 7.94 & -11.11 & 223.80 & 7.25 \\ 3.80 & -2.26 & 7.25 & 7.36 \end{pmatrix} \cdot 10^{-5}.$$

- High risk investment strategy C:

$$\underline{X_t} - \underline{\mu} = \begin{pmatrix} 0.75 & 0.071 & 0.023 & -0.023 \\ 0.10 & 0.87 & -0.0014 & 0.080 \\ 0.18 & 0.053 & 0.72 & 0.38 \\ -0.037 & 0.21 & -0.0022 & 0.80 \end{pmatrix} \cdot (\underline{X_{t-1}} - \underline{\mu}) + \underline{a_t},$$

where $\underline{X}^0 = (0.0072, 0.040, 0.014, 0.0046)^T$, $\underline{\mu} = (0.013, 0.030, 0.047, 0.030)^T$ and the covariance matrix for $\underline{a_t}$ is

$$\underline{\Sigma} = \begin{pmatrix} 10.41 & -1.45 & 11.41 & 3.80 \\ -1.45 & 5.20 & -16.18 & -2.26 \\ 11.41 & -16.18 & 557.40 & 7.52 \\ 3.81 & -2.26 & 7.52 & 7.36 \end{pmatrix} \cdot 10^{-5}.$$

From the estimated parameters we can see that in all three investment strategies, the white noise term for the inflation rate moves in opposite direction with all other three variables which corresponds to the data trend shown in Figure 3.1. This negative correlation can be explained intuitively: When the CPI grows rapidly, the contemporary real investment return and real wage increase are usually impaired by the inflation since the purchasing power for \$1 is weakened. For the covariance matrix $\underline{\Sigma}$, the estimation results show that investment strategy C has the highest volatility with the highest variance of $a_{t,3}$; strategy B as the second highest variance and strategy A as the lowest one.

Having obtained the basic time series model for this study, we will use those three models to carry on analysis of the loss function in the following chapters. The impact of different investment strategies will also be studied.

# Chapter 4

# Stochastic Analysis of the Loss Function

## 4.1 Loss function for hybrid pension plan

Now let us consider the loss function of the plan sponsor for each individual policy at issue. According to the cash flows discussed earlier, we define the loss function as follows:

$$
\begin{aligned}
{}_0L &= \text{Present Value at age 25 of cash flows before retirement} \\
&\quad + \text{Present Value at age 25 of cash flows after retirement} \\
&= -\text{Present Value of all contributions} \\
&\quad + \text{Present Value of all benefit payments}
\end{aligned}
$$

$$
\begin{aligned}
&= -c \cdot S(25) + \sum_{t=25}^{63} (-c) \cdot S(25) \cdot e^{\sum_{i=25}^{t} (\alpha_i + f_i) - \sum_{i=25}^{t} (\delta_i + f_i)} \\
&\quad + RR \cdot S(25) \cdot e^{\sum_{i=25}^{64} (\alpha_i + f_i)} \cdot \sum_{t=65}^{w} {}_{t-65}p_{65} \cdot e^{\sum_{i=25}^{t-1} (-l_i - f_i)} \\
&= -c \cdot S(25) \cdot \left(1 + \sum_{t=25}^{63} e^{\sum_{i=1}^{t} (\alpha_i - \delta_i)}\right) \\
&\quad + RR \cdot S(25) \cdot e^{\sum_{i=25}^{64} (\alpha_i - l_i)} \\
&\quad + RR \cdot S(25) \cdot \sum_{t=66}^{w} {}_{t-65}p_{65} \cdot e^{\sum_{i=25}^{64} (\alpha_i - l_i) + \sum_{i=65}^{t-1} (-l_i - f_i)}.
\end{aligned}
$$

Here $w$ is the maximum age in the life table. In the first expression above, contributions are discounted with the returns on the DC account ($\delta_i + f_i$ for year $i$) while future benefit payments are discounted with long term treasury bond returns ($l_i + f_i$ for year $i$). In the second expression we can see that the variables $\alpha_i - \delta_i$, $\alpha_i - l_i$ and $-l_i - f_i$ are involved. Under the Guassian assumption we have made, it follows that those three variables together follow a three dimensional normal distribution for any given time $i$. The survival probability ${}_{t-65}p_{65}$ is calculated by the 2001 CSO life table (see Appendix A).

## 4.2 Change of variables

In this section we will discuss how to simplify the expression for the loss function through linear transformations of the variable $\underline{X_t}$ and appropriate summations of those variables. The following theorem introduces how to obtain the first two conditional moments of the transformed time series variables.

**Theorem 4.1.** *Let* $\underline{Z_t} = A \cdot \underline{X_t}$*, where* $A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \end{pmatrix}$*. Therefore* $\underline{Z_t}$ *is a three-dimensional time series variable* $\underline{Z_t} = (Z_{t,1}, Z_{t,2}, Z_{t,3})$ *defined as follows:*

$$\begin{aligned} Z_{t,1} &= \alpha_t - \delta_t \\ Z_{t,2} &= \alpha_t - l_t \\ Z_{t,3} &= -l_t - f_t. \end{aligned}$$

*The first two conditional moments of this time series can be obtained as follows:*

$$\begin{aligned} E(\underline{Z_t}|\underline{Z}^0) &= A \cdot E(\underline{X_t}|\underline{X}^0) \\ Cov(\underline{Z_t}, \underline{Z_s}^T|\underline{Z}^0) &= A \cdot (Cov(\underline{X_t}, \underline{X_s}^T|\underline{X}^0)) \cdot A^T, \end{aligned} \tag{4.1}$$

*where* $s, t \in \boldsymbol{Z}, s \leq t$*, and* $\underline{X}^0$ *is the initial value of* $\underline{X_t}$ *observed at issue date.*

**Proof**: For the first moment, the result is straightforward. Since $\underline{Z_t} = A \cdot \underline{X_t}$, we have,

$$\begin{aligned} E(\underline{Z_t}|\underline{X}^0) &= E(A \cdot \underline{X_t}|\underline{X}^0) \\ &= A \cdot E(\underline{X_t}|\underline{X}^0). \end{aligned}$$

Next,

$$\begin{aligned} E(\underline{Z_t} \cdot \underline{Z_s}^T|\underline{X}^0) &= E((A \cdot \underline{X_t}) \cdot (A \cdot \underline{X_s})^T|\underline{X}^0) \\ &= E((A \cdot \underline{X_t}) \cdot \underline{X_s^T} \cdot A^T|\underline{X_0}) \\ &= A \cdot E(\underline{X_t} \cdot \underline{X_s}^T|\underline{X}^0) \cdot A^T. \end{aligned}$$

Therefore,

$$\begin{aligned} Cov(\underline{Z_t}, \underline{Z_s}^T|\underline{X}^0) &= E(\underline{Z_t} \cdot \underline{Z_s}^T|\underline{Z}^0) - (E(\underline{Z_t}|\underline{X}^0) \cdot E(\underline{Z_s}|\underline{Z}^0))^T \\ &= A \cdot E(\underline{X_t} \cdot \underline{X_s}^T|\underline{X}^0) \cdot A^T - A \cdot E(\underline{X_t}|\underline{X}^0) \cdot E(\underline{X_s}|\underline{X}^0)^T \cdot A^T \\ &= A \cdot Cov(\underline{X_t}, \underline{X_s}^T|\underline{X}^0) \cdot A^T. \qquad \square \end{aligned}$$

The loss function can be rewritten as:

$$\begin{aligned} {}_0L &= -c \cdot S(25) + \sum_{t=25}^{63} (-c) \cdot S(25) \cdot e^{\sum_{i=25}^{t} Z_{i,1}} \\ &+RR \cdot S(25) \cdot e^{\sum_{i=25}^{64} Z_{i,2}} + RR \cdot S(25) \cdot \sum_{t=66}^{w} {}_{t-65}p_{65} \cdot e^{\sum_{i=25}^{64} Z_{i,2} + \sum_{i=65}^{t-1} Z_{i,3}}. \end{aligned}$$

Note that $\underline{Z_t}$ is also a Guassian process since it is a linear transformation of the observed Guassian process $\underline{X_t}$. Next, to make the expression for the loss function neater, we define a vector $\underline{X_t}$.

Let $\underline{Y_t} = (Y_{t,1}, Y_{t,2})$ be a two dimensional Guassian process where

$$\begin{aligned} Y_{t,1} &= \begin{cases} 0 & t = 25 \\ \sum_{i=25}^{t-1} Z_{i,1} & t > 25 \end{cases} \\ Y_{t,2} &= \begin{cases} \sum_{i=25}^{t-1} Z_{i,2} & t = 25, 26, ..., 64, 65 \\ \sum_{i=25}^{64} Z_{i,2} + \sum_{i=65}^{t-1} Z_{i,3} & t > 65 \end{cases}. \end{aligned}$$

The expression for ${}_0L$ can be simplified to

$$ {}_0L = \sum_{t=25}^{64} -c \cdot S(25) \cdot e^{Y_{t,1}} + RR \cdot S(25) \cdot \sum_{t=65}^{w} {}_{t-65}p_{65} \cdot e^{Y_{t,2}}. \tag{4.2}$$

In this expression we see that investigating the variable $\underline{Y_t}$ will allow us to study the randomness of ${}_0L$ for this combination hybrid plan.

## 4.3 Conditional Moments of the Loss Function

After simplifying the expression, we will try to obtain conditional moments of the loss function to examine the potential average loss on each policy.

### 4.3.1 Cash Flow "Adjustments"

In order to evaluate Equation (4.2), we redefine the cash flows and discount factors of this plan for modeling purposes:

- During the contribution phase, $25 \leq t \leq 64$, we consider constant the annual cash flows of $CF(t) = -c \cdot S(25)$ which are discounted with the factors $e^{Y_{t,1}}$ for present values.
- After retirement where $t \geq 65$, the annual pension benefit cash flow at age $t$ is $CF(t) = RR \cdot S(25) \cdot {}_{t-65}\, p_{65}$ and the corresponding discount factor is $e^{Y_{t,2}}$.

This method of evaluation separates the deterministic terms from the random terms in the loss function. The deterministic part represents the cash flow and the random part is the discount factor. The cash flow $CF(t)$ is not the actual cash flow that occurs in year $t$. Note that this approach is applied simply to help further modeling, and it has no impact on the numerical results for ${}_0L$.

### 4.3.2 Conditional Moments of ${}_0L$

We will see, from Equation (4.3), that the expected value of ${}_0L$ can be obtained from the expected value of a linear combination of many log-normally distributed variables.

Since $\underline{Y_t}$ follows a Guassian process, at any time $t$, $Y_{t,1}$ and $Y_{t,2}$ are both normally distributed. In order to get the conditional moments of ${}_0L$, we need the conditional moments of $\underline{Y_t}$ given $\underline{X}^0$ first.

The first conditional moment of $\underline{Y_t}$ is simply the sum of the conditional means of $\underline{Z_t}$, that is to say,

$$\begin{aligned} E(Y_{t,1}|\underline{X}^0) &= \sum_{i=25}^{t-1} E(Z_{i,1}|\underline{X}^0) \\ E(Y_{t,2}|\underline{X}^0) &= \begin{cases} \sum_{i=25}^{t-1} E(Z_{i,2}|\underline{X}^0), & \text{if } t \leq 65 \\ \sum_{i=25}^{64} E(Z_{i,2}|\underline{X}^0) + \sum_{i=65}^{t-1} E(Z_{i,3}|\underline{X}^0) & \text{if } t > 65 \end{cases} . \end{aligned}$$

As for the second moment of $\underline{Y_t}$, a recursive method is applied when calculate $Cov(\underline{Y_t}, \underline{Y_s}^T|\underline{X}^0)$ to improve efficiency. Here we assume $t \geq s$. Since $Cov(\underline{Y_s}, \underline{Y_t}^T|\underline{X}^0) = (Cov(\underline{Y_t}, \underline{Y_s}^T|\underline{X}^0))^T$, once we fill up the lower triangle of $Cov(\underline{Y_s}, \underline{Y_t}^T|\underline{X}^0)$ with $25 \leq s \leq t \leq w$, the upper triangle is immediately filled in through transposes. Following are the steps we take to calculate the lower triangle of $Cov(\underline{Y_s}, \underline{Y_t}^T|\underline{X}^0)$ where $25 \leq s \leq w$ and $25 \leq t \leq w$:

1. Since $Var(\underline{Y_{25}}|\underline{X}^0) = 0$, we start by calculating $Var(\underline{Y_{26}}|\underline{X}^0)$.

2. Calculate the first column which represents $Cov(\underline{Y_t}, \underline{Y_{26}}|\underline{X}^0)$ by recursion:

$$Cov(\underline{Y_t}, \underline{Y_{26}}^T|\underline{X}^0) = Cov(\underline{Y_{t-1}}, \underline{Y_{26}}^T|\underline{X}^0) + \begin{cases} Cov((Z_{t-1,1}, Z_{t-1,2})^T, \underline{Y_{26}}^T|\underline{X}^0) & \text{if } t \leq 65 \\ Cov((Z_{t,1}, Z_{t,3})^T, \underline{Y_{26}}^T|\underline{X}^0) & \text{if } t > 65 \end{cases} .$$

3. Move along each row until reaching the diagonal to fill up the lower triangle recursively as followings: when $t \leq 65$,

$$Cov(\underline{Y_t}, \underline{Y_s}^T|\underline{X}^0) = Cov(\underline{Y_t}, \underline{Y_{s-1}}^T|\underline{X}^0) + \sum_{i=26}^{t-1} Cov((Z_{i,1}, Z_{i,2})^T, (Z_{s-1,1}, Z_{s-1,2})|\underline{X}^0).$$

When $t > 65$ and $s \leq 65$,

$$\begin{aligned} Cov(\underline{Y_t}, \underline{Y_s}^T | \underline{X}^0) &= Cov(\underline{Y_t}, \underline{Y_{s-1}}^T | \underline{X}^0) + \sum_{i=25}^{64} Cov((Z_{i,1}, Z_{i,2})^T, (Z_{s-1,1}, Z_{s-1,2}) | \underline{X}^0) \\ &\quad + \sum_{i=65}^{t-1} Cov((Z_{i,1}, Z_{i,3})^T, (Z_{s-1,1}, Z_{s-1,2}) | \underline{X}^0). \end{aligned}$$

When $t > 65$ and $s > 65$,

$$\begin{aligned} Cov(\underline{Y_t}, \underline{Y_s}^T | \underline{X}^0) &= Cov(\underline{Y_t}, \underline{Y_{s-1}}^T | \underline{X}^0) + \sum_{i=25}^{64} Cov((Z_{i,1}, Z_{i,2})^T, (Z_{s-1,1}, Z_{s-1,3}) | \underline{X}^0) \\ &\quad + \sum_{i=65}^{t-1} Cov((Z_{i,1}, Z_{i,3})^T, (Z_{s-1,1}, Z_{s-1,3}) | \underline{X}^0). \end{aligned}$$

Please note that each entry of this $(w-25)$ by $(w-25)$ matrix is actually a two by two matrix instead of a one-dimensional number. Since $\underline{Y_t}$ follows a bivariate normal distribution, we can directly get the variance of $Y_{t,1}$ and $Y_{t,2}$ from the covariance matrix of $\underline{Y_t}$ given $\underline{X}^0$.

Let us move on to the conditional moments of ${}_0L$ given $\underline{X}^0$. Equation (4.3) gives the formula for the expected value of ${}_0L$:

$$\begin{aligned} E({}_0L | \underline{X}^0) &= \sum_{t=25}^{64} -c \cdot S(25) \cdot e^{E(Y_{t,1}|\underline{X}^0) + 0.5 \cdot Var(Y_{t,1}|\underline{X}^0)} \\ &\quad + RR \cdot S(25) \cdot \sum_{i=65}^{w} {}_{t-65}p_{65} \cdot e^{E(Y_{t,2}|\underline{X}^0) + 0.5 \cdot Var(Y_{t,2}|\underline{X}^0)}. \end{aligned} \tag{4.3}$$

Calculations for the second conditional moments of ${}_0L$ given $\underline{X}^0$ are more complex. Parker (1997) introduced how to use recursion on future cash flows for this calculation, and we will follow the same approach here.

Let ${}_0L_k$ denote the present value at issue date for all cash flows until age $k$. Since the first cash flow $c \cdot S(25)$ occurs at issue date, we have

$$E({}_0{L_{25}}^2|\underline{X}^0) = (c \cdot S(25))^2.$$

For $k > 25$, if $k \leq 64$, we have

$$\begin{aligned} {}_0L_k &= {}_0L_{k-1} + CF(k) \cdot e^{Y_{k,1}} \\ &= {}_0L_{k-1} - c \cdot S(25) \cdot e^{Y_{k,1}}. \end{aligned}$$

When $k > 64$,

$$\begin{aligned} {}_0L_k &= {}_0L_{k-1} + CF(k) \cdot e^{Y_{k,2}} \\ &= {}_0L_{k-1} + RR \cdot S(25) \cdot {}_{k-65}\,p_{65} \cdot e^{Y_{k,2}}. \end{aligned}$$

Therefore for $25 < k \leq 64$ we have

$$\begin{aligned} E({}_0L_k^2|\underline{X}^0) &= E(({}_0L_{k-1} + CF(k) \cdot e^{Y_{k,1}})^2|\underline{X}^0) \\ &= E({}_0L_{k-1}^2|\underline{X}^0) + 2 \cdot CF(k) \cdot E({}_0L_{k-1} \cdot e^{Y_{k,1}}|\underline{X}^0) \\ &\quad + CF(k)^2 \cdot E(e^{2 \cdot Y_{k,1}}|\underline{X}^0). \end{aligned} \tag{4.4}$$

In Equation (4.4), $E({}_0L_{k-1}^2|\underline{X}^0)$ is already available from earlier calculations, and

$$\begin{aligned} E({}_0L_{k-1} \cdot e^{Y_{k,1}}|\underline{X}^0) &= E\left(\sum_{i=25}^{k-1} CF(i) \cdot e^{Y_{i,1}} \cdot e^{Y_{k,1}} \middle| \underline{X}^0\right) \\ &= E\left(\sum_{i=25}^{k-1} CF(i) \cdot e^{Y_{i,1}+Y_{k,1}} \middle| \underline{X}^0\right) \\ &= \sum_{i=25}^{k-1} CF(i) \cdot E(e^{Y_{i,1}+Y_{k,1}}|\underline{X}^0) \\ &= \sum_{i=25}^{k-1} CF(i) \cdot e^{E(Y_{i,1}+Y_{k,1}|\underline{X}^0)+1/2 \cdot Var(Y_{i,1}+Y_{k,1}|\underline{X}^0)}, \end{aligned}$$

where $E(Y_{i,1}+Y_{k,1}|\underline{X}^0)$ and $Var(Y_{i,1}+Y_{k,1}|\underline{X}^0)$ can both be obtained as discussed earlier from the moments of $\underline{Y_t}$. The last term in Equation (4.4) is also easily calculated since $e^{2\cdot Y_{k,1}}$ is log-normally distributed.

When $k>64$, similarly we have

$$\begin{aligned}
E({}_0L_k^2|\underline{X}^0) &= E(({}_0L_{k-1}+CF(k)\cdot e^{Y_{k,2}})^2|\underline{X}^0) \\
&= E({}_0L_{k-1}^2|\underline{X}^0)+2\cdot CF(k)\cdot E({}_0L_{k-1}\cdot e^{Y_{k,2}}|\underline{X}^0) \\
&\quad +CF(k)^2\cdot E(e^{2\cdot Y_{k,2}}|\underline{X}^0) \qquad (4.5)
\end{aligned}$$

and

$$\begin{aligned}
E({}_0L_{k-1}\cdot e^{Y_{k,2}}|\underline{X}^0) &= E\left(\sum_{i=25}^{64}CF(i)\cdot e^{Y_{i,1}}\cdot e^{Y_{k,2}}\middle|\underline{X}^0\right)+E\left(\sum_{i=65}^{k-1}CF(i)\cdot e^{Y_{i,2}}\cdot e^{Y_{k,2}}\middle|\underline{X}^0\right) \\
&= E\left(\sum_{i=25}^{64}CF(i)\cdot e^{Y_{i,1}+Y_{k,2}}\middle|\underline{X}^0\right)+E\left(\sum_{i=65}^{k-1}CF(i)\cdot e^{Y_{i,2}+Y_{k,2}}\middle|\underline{X}^0\right) \\
&= \sum_{i=25}^{64}CF(i)\cdot E(e^{Y_{i,1}+Y_{k,2}}|\underline{X}^0)+\sum_{i=65}^{k-1}CF(i)\cdot E(e^{Y_{i,2}+Y_{k,2}}|\underline{X}^0) \\
&= \sum_{i=25}^{64}CF(i)\cdot e^{E(Y_{i,1}+Y_{k,2}|\underline{X}^0)+1/2\cdot Var(Y_{i,1}+Y_{k,2}|\underline{X}^0)} \\
&\quad +\sum_{i=65}^{k-1}CF(i)\cdot e^{E(Y_{i,2}+Y_{k,2}|\underline{X}^0)+1/2\cdot Var(Y_{i,2}+Y_{k,2}|\underline{X}^0)}.
\end{aligned}$$

After obtaining $E({}_0L_k^2|\underline{X}^0)$ for $25\le k\le w$, the conditional mean and variance of ${}_0L$ is calculated as follows:

$$\begin{aligned}
E({}_0L|\underline{X}^0) &= E({}_0L_w|\underline{X}^0) \\
Var({}_0L|\underline{X}^0) &= Var({}_0L_w|\underline{X}^0) \\
&= E({}_0L_w^2|\underline{X}^0)-(E({}_0L_w|\underline{X}^0))^2.
\end{aligned}$$

To investigate the distribution of ${}_0L$ more carefully, we also calculate the third conditional moment of ${}_0L$ which can also be derived using a recursive method. For the first cash flow, we have

$$E({}_0L_{25}^3) = (c \cdot S(25))^3.$$

For $25 < k \leq 64$,

$$\begin{aligned}
E({}_0L_k^3|\underline{X}^0) &= E(({}_0L_{k-1} + CF(k) \cdot e^{Y_{k,1}})^3|\underline{X}^0) \\
&= E({}_0L_{k-1}^3 + 3 \cdot {}_0L_{k-1}^2 \cdot CF(k) \cdot e^{Y_{k,1}} \\
&\quad +3 \cdot {}_0L_{k-1} \cdot (CF(k) \cdot e^{Y_{k,1}})^2 + (CF(k) \cdot e^{Y_{k,1}})^3|\underline{X}^0) \\
&= E({}_0L_{k-1}^3|\underline{X}^0) + 3 \cdot E({}_0L_{k-1}^2 \cdot CF(k) \cdot e^{Y_{k,1}}|\underline{X}^0) \\
&\quad +3 \cdot E({}_0L_{k-1} \cdot (CF(k) \cdot e^{Y_{k,1}})^2|\underline{X}^0) + E((CF(k) \cdot e^{Y_{k,1}})^3|\underline{X}^0).
\end{aligned}$$

In the above expression, $E({}_0L_{k-1}^3|\underline{X}^0)$ is obtained in the previous recursions and

$$E({}_0L_{k-1}^2 \cdot CF(k)e^{Y_{k,1}}|\underline{X}^0) = E\left(\sum_{i=25}^{k-1}\sum_{j=25}^{k-1} CF(i) \cdot CF(j) \cdot CF(k)e^{Y_{i,1}+Y_{j,1}+Y_{k,1}}\middle|\underline{X}^0\right). \tag{4.6}$$

As we know, $Y_{i,1}+Y_{j,1}+Y_{k,1}$ given $\underline{X}^0$ is also normally distributed for any integer $i, j, k$ in $(25, 64]$. It is straightforward that $e^{Y_{i,1}+Y_{j,1}+Y_{k,1}}$ given $\underline{X}^0$ follows a log-normal distribution. Therefore the expected value of each component in the double summation in Equation (4.6) can be calculated.

On the other hand,

$$\begin{aligned}
E({}_0L_{k-1} \cdot (CF(k) \cdot e^{Y_{k,1}})^2|\underline{X}^0) &= E\left(\left(\sum_{i=25}^{k-1} CF(i) \cdot e^{Y_{i,1}}\right) \cdot CF(k)^2 \cdot e^{2Y_{k,1}}\middle|\underline{X}^0\right) \\
&= \sum_{i=25}^{k-1} E\left(CF(i)CF(k)^2 \cdot e^{Y_{i,1}+2Y_{k,1}}\middle|\underline{X}^0\right).
\end{aligned}$$

Therefore $E({}_0L_k^3|\underline{X}^0)$ can then be obtained when $25 \leq k \leq 64$.

For $k > 64$,

$$\begin{aligned}
E({}_0L_k^3|\underline{X}^0) &= E(({}_0L_{k-1} + CF(k) \cdot e^{Y_{k,2}})^3|\underline{X}^0) \\
&= E({}_0L_{k-1}^3|\underline{X}^0) + 3 \cdot E({}_0L_{k-1}^2 \cdot CF(k) \cdot e^{Y_{k,2}}|\underline{X}^0) \\
&\quad +3 \cdot E({}_0L_{k-1} \cdot (CF(k) \cdot e^{Y_{k,2}})^2|\underline{X}^0) + E((CF(k) \cdot e^{Y_{k,2}})^3|\underline{X}^0).
\end{aligned}$$

Here,

$$\begin{aligned}
{}_0L_{k-1}^2 &= \left(\sum_{i=25}^{64} CF(i) \cdot e^{Y_{i,1}} + \sum_{j=65}^{k-1} CF(j) \cdot e^{Y_{j,2}}\right)^2 \\
&= \sum_{i=25}^{64}\sum_{j=25}^{64} CF(i)CF(j) \cdot e^{Y_{i,1}+Y_{j,1}} + 2 \cdot \sum_{i=25}^{65}\sum_{j=65}^{k-1} CF(i)CF(j) \cdot e^{Y_{i,1}+Y_{j,2}} \\
&\quad + \sum_{i=65}^{k-1}\sum_{i=65}^{k-1} CF(i)CF(j) \cdot e^{Y_{i,2}+Y_{j,2}}.
\end{aligned}$$

Using the properties of the log-normal distribution, the third conditional moment of ${}_0L_k$ given $\underline{X}^0$ can be derived by this recursion. The skewness of ${}_0L$ can be calculated as

$$\begin{aligned}
Skew({}_0L|\underline{X}^0) &= \frac{E(({}_0L - E({}_0L|\underline{X}^0))^3|\underline{X}^0)}{Var({}_0L|\underline{X}^0)^{3/2}} \\
&= \frac{E({}_0L^3|\underline{X}^0) - 3 \cdot E({}_0L^2|\underline{X}^0) \cdot E({}_0L|\underline{X}^0) + 2 \cdot (E({}_0L|\underline{X}^0))^3}{Var({}_0L|\underline{X}^0)^{3/2}}.
\end{aligned}$$

Some numerical examples will be provided in the following section based on the VAR(1) models that were estimated in the previous chapter and the formula we just derived.

## 4.4 Numerical Illustrations

In this section we will present results for the conditional moments of the loss function ${}_0L$ given $\underline{X}^0$ with the VAR(1) models we obtained earlier for the investment strategies. Comparison between different investment strategies will be made to investigate the impact of the asset allocation on the investment risk for our combination hybrid pension plan. In the following discussion, we set the first year salary $S(25)$ to 1 and the replacement ratio guarantee is 70%.

### 4.4.1 Deterministic Versus Stochastic Assumptions

First, let us check the impact of the stochastic assumptions on variable $\underline{X_t}$ into this combination hybrid pension plan. In some earlier studies on pension plans, deterministic assumption for $\underline{X_t}$ is used, i.e. $\underline{X_t}$ is assumed to be a constant over time. We will compare our current stochastic assumptions with a deterministic assumption. The basic assumptions for both cases are:

- **Deterministic Assumptions**

  Assume that there are no randomness embedded in the variable $\underline{X_t}$ and that it does not vary through time. The sample means of $\underline{X_t}$ for strategies A, B, and C are chosen as the best estimates for $\underline{X_t}$ and remain the same through time in the deterministic valuation. Here we use investment strategy B which represents the median risk asset allocation as an example. By setting the loss function at issue , ${}_0L$, to zero, we back solve for a contribution rate, which is 14.12% in this case.

- **Stochastic Assumptions**

  Next we use this contribution rate of 14.12% together with the stochastic assumption of the VAR(1) model to obtain the conditional moments of ${}_0L_k$ given $\underline{X}^0$. Therefore with the same contribution rate, and same plan feature, we can identify the difference caused by the stochastic model by comparing these two loss functions.

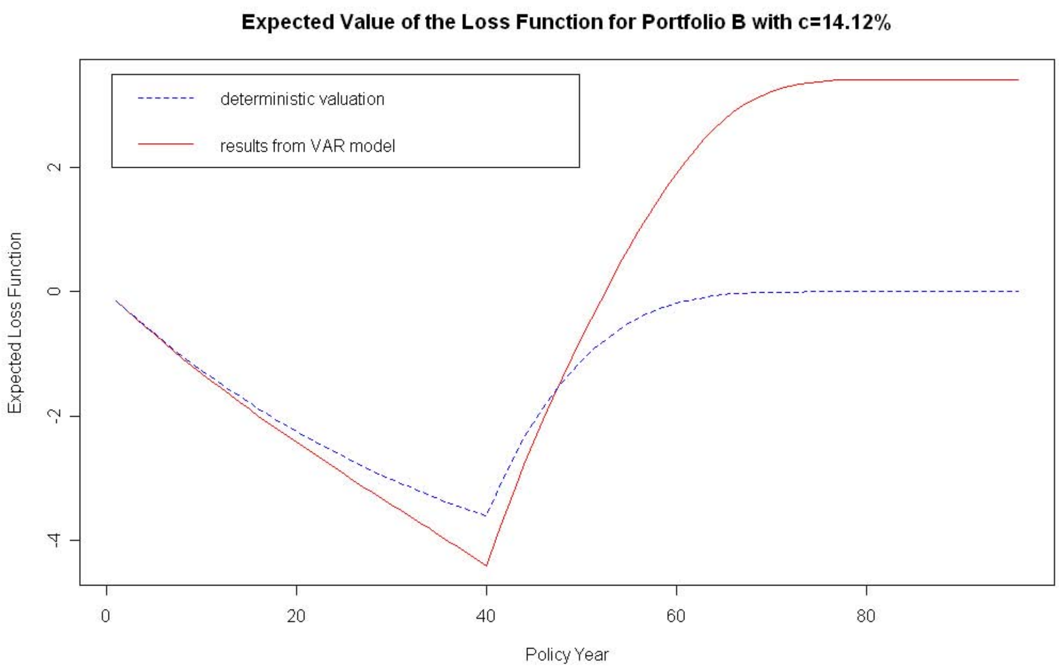


Figure 4.1: Comparison of loss functions on deterministic and stochastic basis

Figure 4.1 compares the expected loss function $E({}_0L_k)$ for the stochastic model and the loss function ${}_0L_k$ valued on a deterministic basis. The x-axis represents the number of years that one has been in this pension plan, i.e., $k - 25$. The graph shows that there are significant differences between the expected loss function and best estimate loss function which should be caused by the change of assumptions in variable $\underline{X_t}$.

We could see that during the contribution phase of this pension plan, the expected value of the DC account which is $-E({}_0L_k|\underline{X}^0)$ under the VAR(1) model is higher than the deterministic $-{}_0L_k$. When positive cash flows, which are the pension payments start, the expected loss function $E({}_0L_k|\underline{X}^0)$ goes up much faster and exceeds the deterministic ${}_0L_k$ after around 5 years. This can be partly explained by the property of log-normal distribution we used for the discount factor. That is

$$E(e^{Y_{t,i}}|\underline{X}^0) = e^{E(Y_{t,i}|\underline{X}^0)+0.5\cdot Var(Y_{t,i}|\underline{X}^0)} > e^{E(Y_{t,i}|\underline{X}^0)} \text{ for } i \in \{1,2\}, t \in \{25, 26, ...w\}.$$

It is also explained by the starting conditions, $\underline{X}^0$ which in this illustration are lower than the mean, $\underline{\mu}$. Therefore the stochastic model not only introduces volatility in ${}_0L$ but also recognizes the current financial situation. The contribution rate obtained on a deterministic basis is insufficient in our stochastic modeling exercise. Hence, since the nature of variable $\underline{X_t}$ is random and starting at $\underline{X}^0$, it is of great importance for the plan sponsor to take these facts into account when investigating the loss function ${}_0L$, and then price this combination hybrid pension plan very carefully.

### 4.4.2 Comparison of Different Investment Strategies

As mentioned in Chapter Three, we consider three types of plan members with different risk appetite. Here we will compare those three investment strategies to find out how they could affect the loss function. Following is a brief review of their investment strategies in the DC account:

- Portfolio A: 20% stock and 80% one-year treasury bond, the low risk portfolio.

- Portfolio B: 50% stock and 50% one-year treasury bond, the median risk portfolio.
- Portfolio C: 80% stock and 20% one-year treasury bond, the high risk portfolio.

With the VAR(1) model we estimated in Chapter Three and the recursive formula that are derived in this chapter, we obtained the conditional moments of ${}_0L_k$, $k \in \mathbf{N}$, for the three investment strategies studied earlier. To make parallel comparisons across all investment strategies, we set the contribution rate to a constant of 14.12% over all portfolios and the replacement ratio guarantee to 70%. Detailed results on the first three moments of ${}_0L_k$ given $\underline{X}^0$ are shown in Table 4.1-4.3.

| Policy Year | | $E({}_0L_k \mid \underline{X}^0)$ | |
|---|---|---|---|
| $(k-25)$ | Strategy A | Strategy B | Strategy C |
| 5 | -0.693 | -0.687 | -0.687 |
| 10 | -1.347 | -1.315 | -1.308 |
| 20 | -2.551 | -2.428 | -2.414 |
| 30 | -3.646 | -3.430 | -3.476 |
| 40 | -4.667 | -4.409 | -4.668 |
| 50 | -1.021 | -0.747 | -1.000 |
| 60 | 1.594 | 1.908 | 1.665 |
| 70 | 2.871 | 3.228 | 2.996 |
| 80 | 3.052 | 3.421 | 3.192 |
| 90 | 3.055 | 3.424 | 3.196 |
| $w$ | 3.055 | 3.424 | 3.196 |

Table 4.1: Expected value of the loss function ${}_0L_k$ throughout the policy term for different investment strategies

Rationally, with sufficient contribution rate, the loss function ${}_0L$ studied here for the plan sponsor should have at most a zero expected value, otherwise the plan sponsor is almost doomed to lose money by offering such a pension plan. However, here for the expected value of ${}_0L_k$ given $\underline{X}^0$, we can see in Table 4.1 that the contribution rate for all three investment strategies are not sufficient since the expect values of the loss function are all quite large given an annual contribution rate of 14.12%. For investment strategy A which has a lower investment risk than strategy B, the expected value of ${}_0L_k$ for $25 \leq k \leq w$ for strategy A is always lower . However, between investment strategies B and C, there are no uniform ordering of $E({}_0L_k|\underline{X}^0)$ for $25 \leq k \leq w$. The expected value of ${}_0L$ given $\underline{X_t}$ for strategy B is actually greater than that of strategy C even though it has a lower portion of stock in the asset allocation. This is because we are comparing the expected value of two variables that are both mixtures of log-normally distributed random variables. For log-normal distributed variables, both the mean and the variance of the corresponding normally distributed variables need to be taken into account when calculating the expected values. It is not clear that a higher variance in the normally distributed variable would give higher expected value of the log-normal distribution if the mean also changes.

The results for the standard deviation and skewness of ${}_0L_k$ are much more straightforward to interpret. As the portion of stock increases in the asset allocation, from strategy A to C, both the standard deviation and the skewness of ${}_0L$, i.e. ${}_0L_w$, go up. Therefore special attention needs to be paid to the portion of stock in the asset allocation of the DC account, because the standard deviation and skewness are rather high compared to the expected value of ${}_0L$. There is a huge investment risk embedded in offering the replacement ratio guarantee as the plan sponsor.

Another thing worth noticing is that $\sigma({}_0L_k|\underline{X}^0)$ increases with policy year. When we are standing at the issue date of the policy and looking forward, as time goes on after the issue date, more and more uncertainty is included in the policy. Therefore

| Policy Year | $\sigma({}_0L_k \mid \underline{X}^0)$ | | |
|---|---|---|---|
| $(k-25)$ | Strategy A | Strategy B | Strategy C |
| 5 | 0.024 | 0.054 | 0.089 |
| 10 | 0.078 | 0.172 | 0.282 |
| 20 | 0.270 | 0.554 | 0.907 |
| 30 | 0.599 | 1.184 | 2.047 |
| 40 | 1.075 | 2.180 | 4.420 |
| 50 | 2.964 | 2.929 | 4.115 |
| 60 | 9.400 | 9.317 | 9.307 |
| 70 | 17.364 | 17.867 | 17.824 |
| 80 | 20.218 | 21.119 | 21.177 |
| 90 | 20.329 | 21.256 | 21.322 |
| $w$ | 20.330 | 21.256 | 21.322 |

Table 4.2: Standard deviation of the loss function ${}_0L_k$ throughout the policy term for different investment strategies

the standard deviation of ${}_0L_k$ is increasing in $k$. As for $Skew({}_0L_k|\underline{X}^0)$, when the cash flows are negative, the skewness is always negative and decreases with time. When the cash flows turn positive, the skewness of ${}_0L_k$ also turns positive after a few years and increases gradually.

Given that a contribution rate of 14.12% is insufficient for all three investment strategies, we now investigate the appropriate contribution rate for each strategy. By setting $E({}_0L) = 0$, the required contribution rate is obtained by trial and error. The results are shown in Table 4.4.

Here we see that the contribution rate is rather high for all three strategies. It is in fact higher than 20% which is not common in practice for regular DB or DC pension plans. This combination hybrid pension plan seems much more "expensive" than regular plans in the current financial context. Among all three investment strategies,

| Policy Year | $Skew({}_0L_k \mid \underline{X}^0)$ | | |
|---|---|---|---|
| $(k-25)$ | Strategy A | Strategy B | Strategy C |
| 5 | -0.150 | -0.337 | -0.560 |
| 10 | -0.236 | -0.534 | -0.901 |
| 20 | -0.466 | -1.009 | -1.764 |
| 30 | -0.778 | -1.736 | -3.621 |
| 40 | -1.155 | -2.993 | -10.189 |
| 50 | 5.188 | 3.777 | -6.158 |
| 60 | 19.419 | 21.106 | 19.403 |
| 70 | 151.107 | 178.612 | 189.291 |
| 80 | 595.377 | 786.544 | 859.120 |
| 90 | 699.437 | 952.772 | 1,049.179 |
| $w$ | 699.574 | 953.031 | 1,049.488 |

Table 4.3: Skewness of the loss function ${}_0L_k$ throughout the policy term for different investment strategies

| Investment Strategy | Contribution Rate | $E({}_0L \mid \underline{X}^0)$ | $\sigma({}_0L \mid \underline{X}^0)$ | $Skew({}_0L_k \mid \underline{X}^0)$ |
|---|---|---|---|---|
| A | 23.361% | 0.000 | 20.154 | 717.704 |
| B | 25.083% | 0.000 | 20.981 | 990.013 |
| C | 23.784% | 0.000 | 21.311 | 1,048.788 |

Table 4.4: Moments for policies with contribution rates adjusted so that $E({}_0L|X_0) = 0$

B requires the highest contribution rate which corresponds with the fact that with the same contribution rate, B gives the biggest expected loss.

We can also find in Table 4.4 that even with different contribution rates, the standard deviation and skewness of ${}_0L$ given $\underline{X}^0$ still increase with the portion of stock in the investment strategy and they are still incredibly high compared to the contribution rate based on the best estimate.

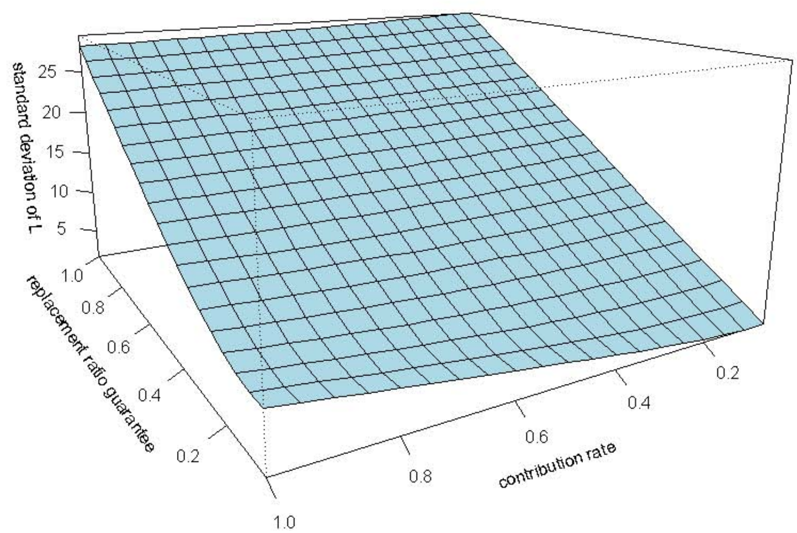


Figure 4.2: Standard deviation of ${}_0L$ for investment strategy A

Looking at the large numbers for the standard deviation and skewness of ${}_0L$, we wonder if it is possible to dampen some of the market risks by changing parameters such as the contribution rate and the replacement ratio guarantees. Figures 4.2-4.4 illustrate how the standard deviation of ${}_0L$ would change with both the contribution rate and the replacement ratio guarantee for all three investment strategies. In those graphs the contribution rate varies from 10% to 100% and the replacement ratio also varies in this range. Though the contribution might never be as high as 100% and the replacement ratio should not fall too low in practice, we are only applying this range to investigate the changes in $\sigma({}_0L|\underline{X}^0)$.

First we find in the graphs that the maximum value of $\sigma({}_0L|\underline{X}^0)$ in each graph increases with the portion of stocks in the investment strategies. Looking into each

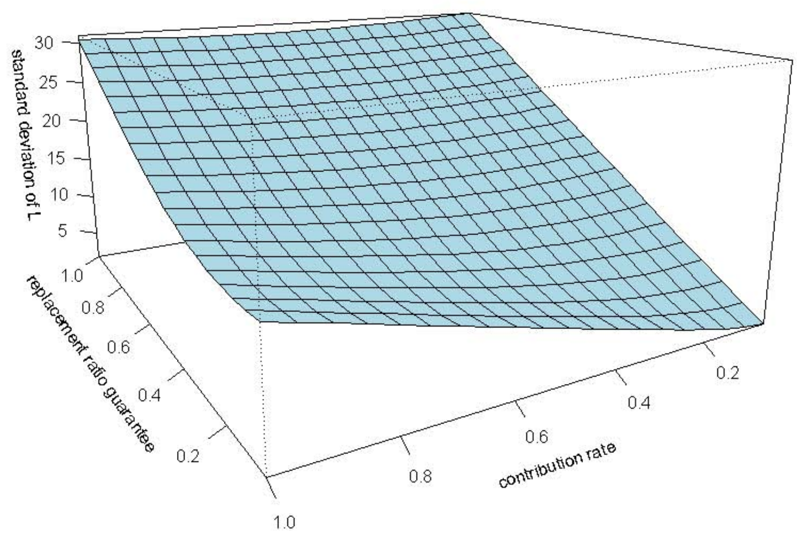


Figure 4.3: Standard deviation of ${}_0L$ for investment strategy B

graph, we see that the surfaces of $\sigma({}_0L|\underline{X}^0)$ for different investment strategies show some similarity and yet slight differences in their shapes. For all three strategies, $\sigma({}_0L|\underline{X}^0)$ always increases in the replacement ratio guarantee. This is due to the fact that the higher the replacement ratio guarantee is, the more pension benefit is guaranteed by the plan sponsor and therefore the sponsor bears more responsibility to undertake risks. When it comes to the contribution ratio, $\sigma({}_0L|\underline{X}^0)$ for all three investment strategies increase with the contribution rate for relatively small replacement ratio guarantee. However, for larger values of replacement ratio guarantee, the trend varies through strategies. For strategy A, $\sigma({}_0L|\underline{X}^0)$ decreases with the contribution rate for high replacement ratio guarantees. While for strategies B and C, the change in $\sigma({}_0L|\underline{X}^0)$ with the contribution rate for a high replacement ratio guarantee is not monotone. Taking the range of acceptable replacement ratio into account, we

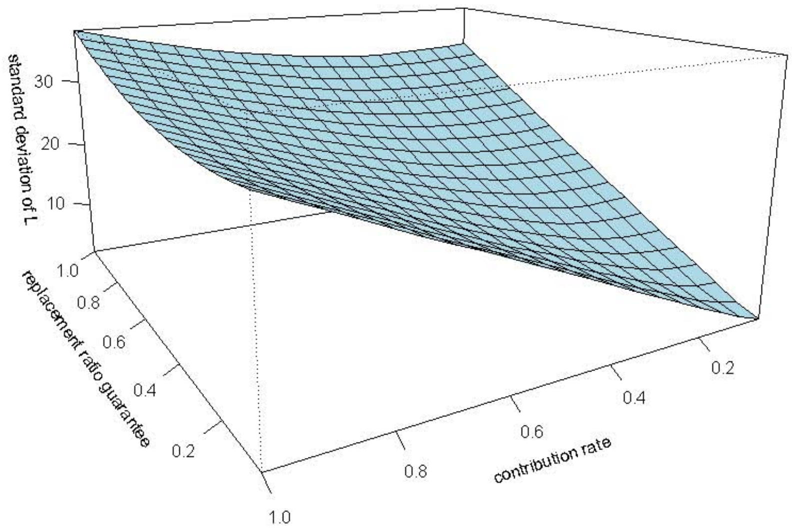


Figure 4.4: Standard deviation of ${}_0L$ for investment strategy C

can find that all three surfaces are quite flat over the desired values of the replacement ratio guarantee and no significant decrements are seen by changing both the contribution and the replacement ratio within the range we discussed before.

So far we can conclude that in order to provide a competitive plan with sufficient pension income for its plan members, which means that the replacement ratio should be set in the range from 60% to 74% as discussed earlier, the extend to which we can dampen the volatility in ${}_0L$ through adjustments of those two parameters is quite limited based on the relatively flat shape of the surface. The contribution rate also needs to be set at an acceptable value for plan members, which limit the dampening even more.

It seems that it would be better for the plan sponsor to focus on choosing a proper investment strategy and then set the contribution rate under the sponsor's own risk tolerance. Very careful investigation of ${}_0L$ is needed since the standard deviation we obtained based on current market data is quite high and can result in significant loss to the plan sponsor if this combination hybrid pension plan is not properly priced and valued.

We will further investigate the distribution of ${}_0L$ under each investment strategies and introduce how risk management can be performed on this hybrid pension plan in the following chapters.

# Chapter 5

# Distribution of the Loss Function

In the previous chapter we derived the conditional moments of the loss function for one individual policy from a limiting portfolio of a combination hybrid pension plan. The first three conditional moments were calculated recursively from the date of issue. From Equation (4.3) we see that ${}_0L$ is a mixture of correlated log-normal random variables. Since the standard deviation and skewness of ${}_0L$ given $\underline{X}^0$ are quite high, it is difficult to fit the entire conditional distribution by matching the first three conditional moments. We will try to estimate and simulate the distribution function of ${}_0L$ in this chapter to gain more insight.

## 5.1 Approximation

Parker(1993a) introduced an iterative approximation method to obtain the distribution of the present value of future cash flows when the discount factor is one-dimensional. Here we will extend this method to two-dimensional discount factors $\underline{Y_t}$

and seek for the conditional distributions of ${}_0L$.

First let us recall the simplified expression for ${}_0L$:

$$ {}_0L = \sum_{t=25}^{64} (-c) \cdot S(25) \cdot e^{Y_{t,1}} + RR \cdot S(25) \cdot \sum_{t=65}^{w} {}_{t-65}p_{65} \cdot e^{Y_{t,2}}. $$

We further simplify the expression by setting $S(25) = 1$, therefore

$$ {}_0L = \sum_{t=25}^{64} (-c) \cdot e^{Y_{t,1}} + RR \cdot \sum_{t=65}^{w} {}_{t-65}p_{65} \cdot e^{Y_{t,2}}. \tag{5.1} $$

Next we introduce the iterative approximation for the conditional distribution of ${}_0L_k$ for $k \in \{25, 26, ..., w\}$:

**Theorem 5.1.** *Let the function $g_j(p, \underline{y})$ where $j \in \{25, 26, ..., w\}$ be defined as*

$$ g_j(p, \underline{y}) = P({}_0L_j \leq p | \underline{Y_j} = \underline{y}) \cdot f_{\underline{Y_j}}(\underline{y}). \tag{5.2} $$

*The distribution of ${}_0L_j$ can be expressed as*

$$ P({}_0L_j \leq p) = \iint_{R^2} g_j(p, \underline{y}) dy_1 \, dy_2. $$

*An approximation method can be applied to derive $g_j(p, \underline{y})$.*

*For $27 \leq j \leq 64$*

$$ g_j(p, \underline{y}) \doteq \iint_{R^2} g_{j-1}(p - CF(j) \cdot e^{y_1}, \underline{x}) \cdot f_{\underline{Y_j}|\underline{Y_{j-1}}}(\underline{y}|\underline{x}) dx_1 \, dx_2. \tag{5.3} $$

*While $j \geq 65$,*

$$ g_j(p, \underline{y}) \doteq \iint_{R^2} g_{j-1}(p - CF(j) \cdot e^{y_2}, \underline{x}) \cdot f_{Y_j|Y_{j-1}}(y|x) dx_1 \, dx_2, \tag{5.4} $$

*here* $f_{\underline{Y_j}|\underline{Y_{j-1}}}(\underline{y}|\underline{x})$ *is the conditional density function of* $\underline{Y_j}$ *given* $\underline{Y_{j-1}}$.

*The starting value for this iteration is*

$$\begin{aligned} g_{26}(p,\underline{y}) &= P({}_0L_{26} < p|\underline{Y_{26}} = \underline{y}) \cdot f_{\underline{Y_{26}}}(\underline{y}) \\ &= f_{\underline{Y_{26}}}(\underline{y}) \cdot I\{p \geq CF(26) \cdot e^{y_{26,1}} + CF(25)\}, \end{aligned}$$

*where* $I\{\cdot\}$ *is an indicator function.*

**Proof:** The relation between $g_j(p,\underline{y})$ and $P({}_0L_j \leq p)$ which is

$$P({}_0L_j \leq p) = \iint\limits_{R^2} g_j(p,\underline{y}) dy_1\, dy_2$$

is a well know result (see, for example, Morrison (1990),Chapter Three).

For the starting value, it is straightforward that

$$\begin{aligned} g_{26}(p,\underline{y}) &= P\Big({}_0L_{26} \leq p|\underline{Y_{26}} = \underline{y}\Big) \cdot f_{\underline{Y_{26}}}(\underline{y}) \\ &= P\Big(CF(25) + CF(26) \cdot e^{y_1} \leq p|\underline{Y_{26}} = \underline{y}\Big) \cdot f_{\underline{Y_{26}}}(\underline{y}) \\ &= f_{\underline{Y_{26}}}(\underline{y}) \cdot \begin{cases} 0, & \text{if } CF(25) + CF(26) \cdot e^{y_1} > p \\ 1, & \text{if } CF(25) + CF(26) \cdot e^{y_1} \leq p \end{cases}. \end{aligned}$$

Now we justify the approximation. There is an alternative expression for $g_j(p,\underline{y})$ defined in Equation (5.2), that is

$$g_j(p,\underline{y}) = f_{\underline{Y_t}}(\underline{y}|{}_0L_j \leq p) \cdot P({}_0L_j \leq p). \tag{5.5}$$

The equivalency is also presented in Morrison (1990), therefore we have:

$$f_{\underline{Y_j}}(\underline{y}|{}_0L_j \leq p) \cdot P({}_0L_j \leq p) = P({}_0L_j \leq p|\underline{Y_j} = \underline{y}) \cdot f_{\underline{Y_j}}(\underline{y}).$$

So far when $j \leq 64$, $g_j(p, \underline{y})$ can be written as:

$$\begin{aligned} g_j(p, \underline{y}) &= P({}_0L_j \leq p | \underline{Y_j} = \underline{y}) \cdot f_{\underline{Y_j}}(\underline{y}) \\ &= P({}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}} | \underline{Y_j} = \underline{y}) \cdot f_{\underline{Y_j}}(\underline{y}) \\ &= \frac{P({}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}}, \underline{Y_j} = \underline{y})}{P(\underline{Y_j} = \underline{y})} \cdot f_{\underline{Y_j}}(\underline{y}) \\ &= \frac{P(\underline{Y_j} = \underline{y} | {}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}})}{P(\underline{Y_j} = \underline{y})} \\ &\quad \cdot P({}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}}) \cdot f_{\underline{Y_j}}(\underline{y}) \\ &= P({}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}}) \cdot f_{\underline{Y_j}}(\underline{y} | {}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}}). \end{aligned}$$

Here,

$$\begin{aligned} & f_{\underline{Y_j}}(\underline{y} | {}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}}) \\ = & \iint_{R^2} f_{\underline{Y_j} | \underline{Y_{j-1}}}(\underline{y} | {}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}}, \underline{Y_{j-1}} = \underline{x}) \\ & \cdot f_{Y_{j-1}}(\underline{x} | {}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}}) dx_1 \, dx_2. \end{aligned} \tag{5.6}$$

Equation (5.6) uses the definition of a conditional density function,

$$\begin{aligned} & f_{\underline{Y_j} | \underline{Y_{j-1}}}(\underline{y} | {}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}}, \underline{Y_{j-1}} = \underline{x}) \\ = & \frac{f_{\underline{Y_j}, \underline{Y_{j-1}}}(\underline{y}, \underline{x} | {}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}})}{f_{\underline{Y_{j-1}}}(\underline{x} | {}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}})}. \end{aligned}$$

Now we make the core approximation in Equation (5.6) which is

$$f_{\underline{Y_j} | \underline{Y_{j-1}}}(\underline{y} | {}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}}, \underline{Y_{j-1}} = \underline{x}) \doteq f_{\underline{Y_j} | \underline{Y_{j-1}}}(\underline{y} | \underline{x}). \tag{5.7}$$

Next, let us proceed with the derivation of the iteration formula for $g_j(p, \underline{y})$ will and validate the approximation later. Assuming Equation (5.7), we have

$$\begin{aligned} g_j(p,\underline{y}) &\doteq P({}_0L_{j-1} \le p - CF(j) \cdot e^{y_{j,1}}) \\ &\quad \cdot \iint\limits_{R^2} f_{\underline{Y_j}|\underline{Y_{j-1}}}(\underline{y}|\underline{x}) \cdot f_{Y_{j-1}}(\underline{x}|{}_0L_{j-1} \le p - CF(j) \cdot e^{y_{j,1}}) dx_1\, dx_2. \end{aligned}$$

Since

$$g_{j-1}(p - CF(j) \cdot e^{y_{j,1}}, \underline{x}) = f_{Y_{j-1}}(\underline{x}|{}_0L_{j-1} \le p - CF(j) \cdot e^{y_{j,1}}) \cdot P({}_0L_{j-1} \le p - CF(j) \cdot e^{y_{j,1}}),$$

we get

$$\begin{aligned} g_j(p,\underline{y}) &\doteq P({}_0L_{j-1} \le p - CF(j) \cdot e^{y_{j,1}}) \\ &\quad \cdot \iint\limits_{R^2} f_{\underline{Y_j}|\underline{Y_{j-1}}}(\underline{y}|\underline{x}) \cdot \frac{g_{j-1}(p - CF(j) \cdot e^{y_{j,1}}, \underline{x})}{P({}_0L_{j-1} \le p - CF(j) \cdot e^{y_{j,1}})} dx_1\, dx_2 \\ &= \iint\limits_{R^2} f_{\underline{Y_j}|\underline{Y_{j-1}}}(\underline{y}|\underline{x}) \cdot g_{j-1}(p - CF(j) \cdot e^{y_{j,1}}, \underline{x}) dx_1\, dx_2. \end{aligned}$$

After deriving the approximation for the contribution phase where $j \le 64$, we can carry on with the pension income phase where $k > 64$. The main difference here is that for $j > 64$, $g_j(p,\underline{y})$ can be written as:

$$\begin{aligned} g_j(p,\underline{y}) &= P({}_0L_j \le p|\underline{Y_j} = \underline{y}) \cdot f_{\underline{Y_j}}(\underline{y}) \\ &= P({}_0L_{j-1} \le p - CF(j) \cdot e^{y_{j,2}}|\underline{Y_j} = \underline{y}) \cdot f_{\underline{Y_j}}(\underline{y}). \end{aligned}$$

By making a similar approximation as Equation (5.7) for $j > 64$ which is

$$f_{\underline{Y_j}|\underline{Y_{j-1}}}(\underline{y}|{}_0L_{j-1} \le p - CF(j) \cdot e^{y_{j,2}}, \underline{Y_{j-1}} = \underline{x}) \doteq f_{\underline{Y_j}|\underline{Y_{j-1}}}(\underline{y}|\underline{x}), \tag{5.8}$$

we can obtain

$$g_j(p,\underline{y}) \doteq \iint\limits_{R^2} g_{j-1}(p - CF(j) \cdot e^{y_2}, \underline{x}) \cdot f_{Y_j|Y_{j-1}}(y|x) dx_1\, dx_2. \quad \square$$

It is very important to notice that the accuracy of this approximation is depends the correlations between $\underline{Y_j}$ and $\underline{Y_{j-1}}$, as well as $e^{Y_{j,1}}, e^{Y_{j,2}}$ and $-{}_0L_j$. We suggest that the information provided by $\underline{Y_{j-1}} = \underline{x}$ is almost the same as that provided by the combination of $\underline{Y_{j-1}} = \underline{x}$ and ${}_0L_{j-1} \leq p - CF(j) \cdot e^{y_{j,1}}$ when used as the condition in the density function of $\underline{Y_j}$. A high correlation usually holds for $j \leq 64$, however for $j > 64$, close investigation needs to be done before the approximation could be applied.

Note that the variable $Y_{j,2}$ is not actually involved when we apply the recursive method during the contribution phase of this combination hybrid plan, therefore the two-dimensional iteration in Theorem 5.1 can be reduced to a simple one-dimensional iteration with $CF(i) = -c$ and discount factor $e^{Y_{i,1}}$ for $25 \leq i \leq 64$ as discussed in Parker(1993a) if the only interest lies in the contribution phase.

On the computational side, we use a discretization method for the double integral in Equations (5.3) and (5.4). Since we have extended the situation from one-dimensional in Parker(1993a) to a two-dimensional model, the computational error due to the discretization is therefore higher compared to Parker(1993a). In order to provide the same accuracy, the number of points needed in the discretization must be greater than that of the one-dimensional case. Also, the runtime for the iteration increases quadratically from the one-dimensional case. In order to maintain an acceptable accuracy, the number of different points used for $Y_{j,1}$ and $Y_{j,2}$ in the discretization is set by trial and error with the runtime taken into account.

## 5.2 Numerical Illustrations

Next we will present some illustrations based on this approximation method with a simple policy which can help us build a general understanding of the distribution of the loss function ${}_0L$. The reason we do not use a practical policy is that the errors caused by the computations for a practical policy are quite high and would shadow the mathematical accuracy of the approximation. Also, since we are using a bivariate normal distribution for $\underline{Y_t}$, the errors caused by matrix calculations is also quite significant.

Assume that we have a short term combination hybrid pension plan which has 10 years of contributions and 10 years of pension payments. The survival probability is modeled with a linear decrement that hits zero at the end of these 20 years. We ignore the first contribution made at age of entry to the plan since there is no randomness associated with it. First let us look at the results for the first 10 years of this policy with investment strategy A in Table 5.1. The contribution rate is set at 34% in order to make sure that $E({}_0L|\underline{X}^0) = 0$. Results show that the iteration method gives a very close approximation of the expected value of ${}_0L_k$ where the error is within 0.17%. For $\sigma({}_0L_{10}|\underline{X}^0)$, the error is about 2.71%.

We can find validations for this approximation in Table 5.2. Considering that the cash flows are negative for the first 10 policy years, the closer the correlation between ${}_0L_k$ and $e^{Y_{k,1}}$ gets to $-1$, the more accurate this approximation method could be.

Now we move on to the pension income phase. Tables 5.3 and 5.4 give the approximation results with the first two moments of ${}_0L_k$ for investment strategy A when

| | $E({}_0L_k\|\underline{X}^0)$ | | | $\sigma({}_0L_k\|\underline{X}^0)$ | | |
|---|---|---|---|---|---|---|
| Policy Year | Actual | Approxi. | Error(%) | Actual | Approxi. | Error(%) |
| 2 | -0.338 | -0.338 | 0.00% | 0.0096 | 0.0096 | 0.00% |
| 3 | -0.672 | -0.672 | 0.00% | 0.023 | 0.0232 | 0.87% |
| 4 | -1.002 | -1.002 | 0.00% | 0.0396 | 0.0403 | 1.77% |
| 5 | -1.329 | -1.329 | 0.00% | 0.0588 | 0.0604 | 2.72% |
| 6 | -1.652 | -1.652 | 0.00% | 0.0802 | 0.0831 | 3.62% |
| 7 | -1.971 | -1.971 | 0.00% | 0.1039 | 0.1079 | 3.85% |
| 8 | -2.286 | -2.286 | 0.00% | 0.1297 | 0.1347 | 3.86% |
| 9 | -2.597 | -2.598 | 0.04% | 0.1579 | 0.1631 | 3.29% |
| 10 | -2.905 | -2.91 | 0.17% | 0.1885 | 0.1936 | 2.71% |

Table 5.1: Approximation results for the first 10 policy years with $c = 34\%$, $S(25) = 1$ and investment strategy A

| Policy Year | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $\rho({}_0L_k, e^{Y_{k,1}})$ | -1.00 | -0.96 | -0.94 | -0.92 | -0.90 | -0.89 | -0.89 | -0.88 | -0.88 |

Table 5.2: Correlation of ${}_0L_k$ and $e^{Y_{k,1}}$ for the first 10 policy years with $c = 34\%$ and investment strategy A

the number of points used in the discretization is set to 25 and 35 respectively. The precision of the approximation becomes weaker and weaker as time goes on for the expected value of ${}_0L_k$ in both of these tables, while the approximation for the standard deviation of ${}_0L_k$ remains relatively close. When we increase the number of points in the discretization, the results are improved and more accurate. We will further investigate the distribution function in the next section to compare the approximation method with the simulation method. Note that the runtime for the approximation with 25 points is several hours. If we increase the number of points from 25 to 35, the runtime is five times as long.

| | $E({}_0L_k\|\underline{X}^0)$ | | | $\sigma({}_0L_k\|\underline{X}^0)$ | | |
|---|---|---|---|---|---|---|
| Policy Year | Actual | Approxi. | Error(%) | Actual | Approxi. | Error(%) |
| 11 | -2.3 | -2.312 | 1% | 0.186 | 0.192 | 3% |
| 12 | -1.784 | -1.798 | 1% | 0.212 | 0.216 | 2% |
| 13 | -1.348 | -1.366 | 1% | 0.264 | 0.265 | 0% |
| 14 | -0.984 | -1.029 | 5% | 0.331 | 0.327 | 1% |
| 15 | -0.685 | -0.787 | 15% | 0.405 | 0.396 | 2% |
| 16 | -0.447 | -0.624 | 40% | 0.479 | 0.468 | 2% |
| 17 | -0.263 | -0.528 | 101% | 0.546 | 0.533 | 2% |
| 18 | -0.129 | -0.497 | 285% | 0.603 | 0.588 | 2% |
| 19 | -0.043 | -0.524 | 1119% | 0.645 | 0.629 | 2% |
| 20 | -0.001 | -0.606 | 60500% | 0.668 | 0.658 | 1% |

Table 5.3: Approximation results for the last 10 policy years with $c = 34\%$, $S(25) = 1$, 25 points in discretization for investment strategy A

## 5.3 Simulations

As discussed earlier, the quality approximation method for the distribution of ${}_0L$ depends on the correlations between each pairing of $e^{Y_{j,1}}$, $e^{Y_{j,2}}$ and ${}_0L_j$. Also for this 2-dimensional case, both the accuracy and the runtime highly depend on the number of points in the discretization. Therefore we need to seek other methods that can provide insights on the distribution of ${}_0L$ with a high level of accuracy.

Considering the definition of the variable ${}_0L$, we think that it is best to use a multivariate simulation study on $\underline{X_t}$ and ${}_0L_t$ for $t \in \{25, 26, ..., w\}$ to get an empirical distribution of ${}_0L$. To compare with the approximation method, we also use a 20-year

| | $E({}_0L_k\|\underline{X}^0)$ | | | $\sigma({}_0L_k\|\underline{X}^0)$ | | |
|---|---|---|---|---|---|---|
| Policy Year | Actual | Approxi. | Error(%) | Actual | Approxi. | Error(%) |
| 11 | -2.3 | -2.301 | 0% | 0.186 | 0.195 | 5% |
| 12 | -1.784 | -1.785 | 0% | 0.212 | 0.219 | 3% |
| 13 | -1.348 | -1.349 | 0% | 0.264 | 0.269 | 2% |
| 14 | -0.984 | -0.987 | 0% | 0.331 | 0.335 | 1% |
| 15 | -0.685 | -0.706 | 3% | 0.405 | 0.407 | 1% |
| 16 | -0.447 | -0.507 | 13% | 0.479 | 0.479 | 0% |
| 17 | -0.263 | -0.379 | 44% | 0.546 | 0.546 | 0% |
| 18 | -0.129 | -0.310 | 141% | 0.603 | 0.602 | 0% |
| 19 | -0.043 | -0.296 | 588% | 0.645 | 0.642 | 0% |
| 20 | -0.001 | -0.329 | 32755% | 0.668 | 0.664 | 1% |

Table 5.4: Approximation results for the last 10 policy years with $c = 34\%$, $S(25) = 1$, 35 points in discretization for investment strategy A

pension plan which include 10 years of contribution and 10 years of pension payments. Results for the first two moments of ${}_0L_t$ given $\underline{X}^0$ are shown in Table 5.5.

| | | $E({}_0L_k\|\underline{X}^0)$ | | $\sigma({}_0L_k\|\underline{X}^0)$ | |
|---|---|---|---|---|---|
| Strategy | Policy Year | Actual | Simulation | Actual | Simulation |
| A | 10 | -2.905 | -2.905 | 0.186 | 0.189 |
| | 20 | -0.001 | 0.003 | 0.668 | 0.671 |
| B | 10 | -2.910 | -2.910 | 0.425 | 0.425 |
| | 20 | -0.007 | -0.007 | 0.763 | 0.764 |
| C | 10 | -2.900 | -2.899 | 0.700 | 0.700 |
| | 20 | 0.002 | 0.005 | 0.948 | 0.952 |

Table 5.5: Simulation results for all three investment strategies

The number of simulations we performed for each strategy is $200,000$. We can tell from Table 5.5 that simulation closely matches the actual moments of ${}_0L_k$ with chosen number of simulations. Therefore we can also use simulation to examine the accuracy of the iterative approximation method.

## 5.4 Comparison

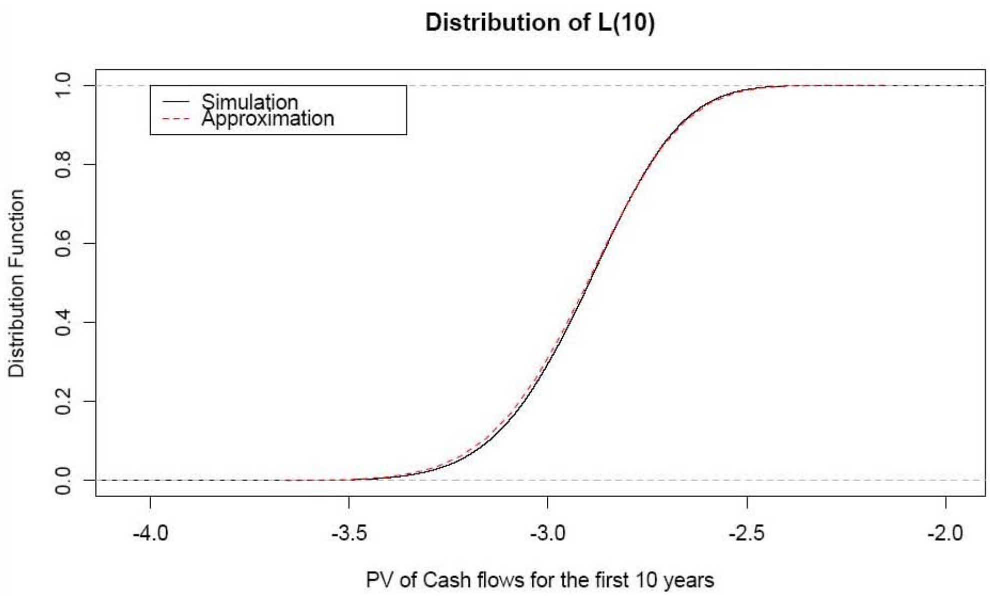


Figure 5.1: Comparison of distribution function for ${}_0L_{10}$ for both methods

Next let us compare the approximation method and the simulation method. Figure 5.1 compares the distribution function we obtain from the approximation method with 25 points in the discretization and the one from the simulation method for ${}_0L_{10}$ with strategy A. They turn out to be very close to each other. After the contribution phase, we compare the simulation results with those for the approximation with different number of points in the discretization. We see in Figure 5.2 that as the number of points increases, the approximation produces closer results to the simulation results. After comparing the moments of ${}_0L$ under both methods, we find that the simulation method provides better results. It is almost like that the curve from the approximation method is parallel but to the left of the simulated curve.

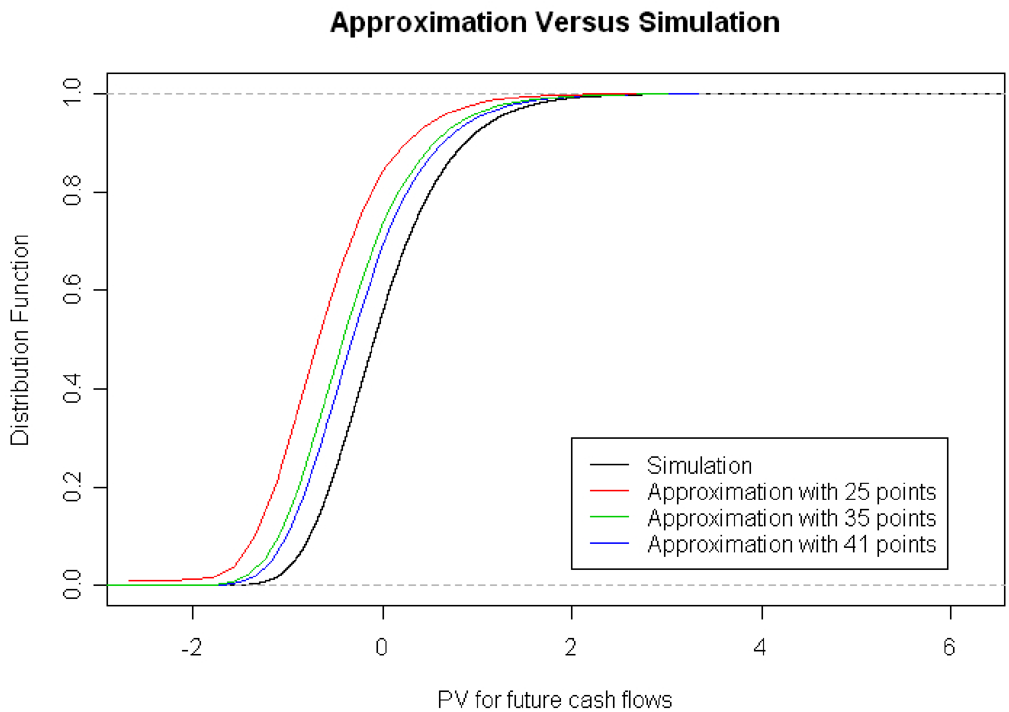


Figure 5.2: Comparison of distribution function for ${}_0L$ for both methods

Next let us compare the impact of different investment strategies in the DC account. From Figure 5.3 which gives the simulated distribution of ${}_0L$ with all investment strategies with different $c$ such that $E({}_0L|\underline{X}^0) = 0$, we find that the riskiness of the investment strategy has a direct impact on ${}_0L$.

For the probability that ${}_0L \leq 0$, strategy A offers the highest value while strategy B gives the lowest one. That is to say, even if the expected values of ${}_0L$ are all set to 0, the probability that the plan sponsor will have a negative loss highly depends on the riskiness of the DC account. Since portfolio C has the highest portion of stock in

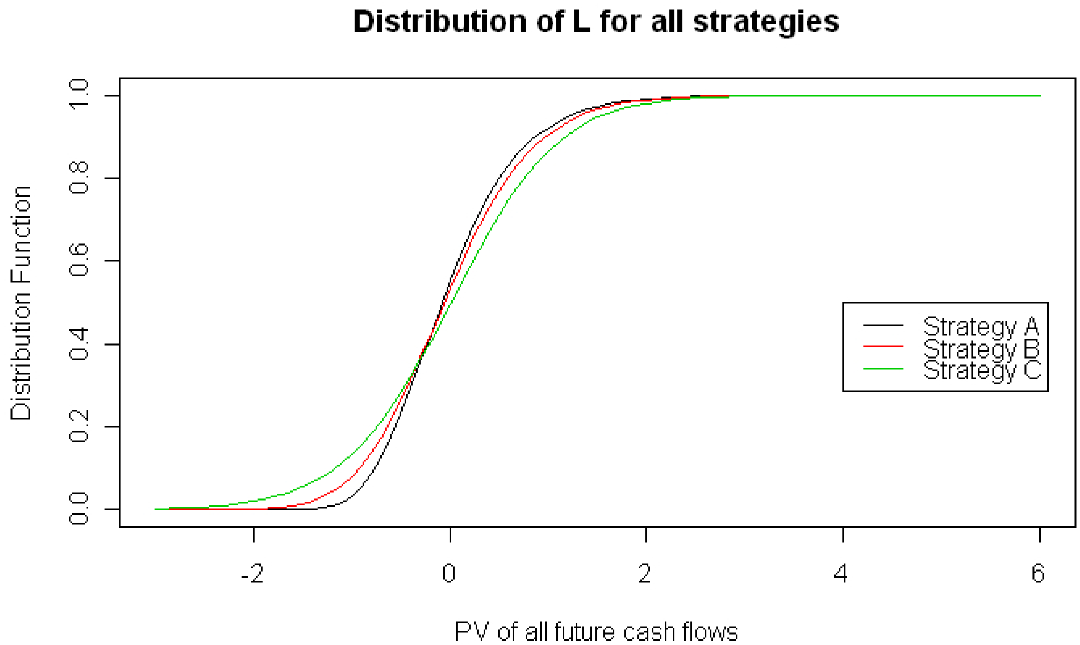


Figure 5.3: Comparison of distribution function for ${}_0L$ for different investment strategies

the DC account and portfolio B has the second highest, we conclude that the higher the portion of stock is, the larger probability is that for the plan sponsor will incur a loss. Therefore, to avoid significant losses, the plan sponsor should be very careful about the asset allocation that is allowed in the DC account.

# Chapter 6

# Economic Capital

Now let us consider a practical application of the results we have so far for this combination hybrid pension plan, which is the calculation of the economic capital for market risk. The concept of Economic Capital has been a great focus for insurance companies in recent years, and it is embedded in the Enterprise Risk Management (ERM) framework which has drawn lots of attention under current market conditions where many of the insurance companies are facing significant losses in their asset investment due to the US Subprime Mortgage Crisis. Although Economic Capital is usually calculated at a corporate level and we are not actually modeling the entire business of an insurance company, we can still take this combination hybrid pension plan as one product sold by an insurance company and thus get the economic capital for market risk associated with this product.

First, we will introduce the properties of risk measure and Value-at-Risk and then the concept of Economic Capital will be introduced in details.

## 6.1 Risk Measures

For insurers, the essence of selling an insurance contract or any type of guarantee is actually selling risk coverage. Therefore it is very important that the insurer knows accurately how to measure risks and then price it profitably or decide the capital requirement to prepare for unexpected losses and stay solvent for a certain time horizon.

There are many types of risk measures that have been applied so far. However, none of them can provide a full coverage of the inherent risk. Each risk measure is usually chosen to focus on the most desired perspective. For instance, the expected value of a random loss gives the central tendency and the variance of the random loss measures the spread.

A general definition for risk measure is as follows (see Denuit, Dhaene, Goovaerts and Kaas 2005):

**Definition 6.1.** *A risk measure is a functional $\rho$ mapping a risk $X$ to a non-negative real number $\rho[X]$, possibly infinite, representing the extra cash flow which has to be added to $X$ to make it acceptable.*

In this definition, the word "acceptable" could be interpreted in many different ways. A risk measure could be the "acceptable" premium that the insurer should charge for an insurance contract, or the "acceptable" amount of risk capital for a portfolio to avoid insolvency. For a given measurement $\rho$, the higher $\rho[X]$ is, the more "risky" $X$ is.

One or more of the following properties are usually desired for a risk measure:

1. **Non-excessive loading:** The risk measure $\rho[X]$ for any random variable $X$ is no greater than the largest possible value of $X$. i.e.

$$\rho[X] \leq max(X) = F_X^{-1}(1)$$

2. **Non-negative loading:** The risk measure $\rho[X]$ for any random variable $X$ is no less than the expected value of $X$. i.e.

$$\rho[X] \geq E[X]$$

3. **Translativity:**

   $\rho[X + c] = \rho[X] + c$ for any random variable $X$ and any constant $c$.

4. **Constancy:**

   For any constant $c$, $\rho[c] = c$

5. **Subadditivity:**

   For any random variable $X$ and $Y$, $\rho[X + Y] \leq \rho[X] + \rho[Y]$

6. **Positive Homogeneity:**

   For any random variable $X$ and any positive constant $c$, $\rho[cX] = c\rho[X]$.

7. **Monotonicity**:

   For any random variable $X$ and $Y$, if $Pr[X \leq Y] = 1$, then $\rho[X] \leq \rho[Y]$.

The first property is quite fundamental since there is no point to buy an insurance contract if the premium is greater than the maximum possible loss, or to keep more risk capital than any possible loss in the portfolio. The situation is similar with the second property if the risk measure $\rho$ represent the premium for an insurance contract.

According to the law of large number, if $\rho[X] < E[X]$, the probability of ruin is 1. The properties listed above are favored by various risk measures based on their own focus. There are some other properties that are considered to be of great importance for risk measures, however no attempt is made here to cover all properties.

## 6.2 Value-at-Risk (VaR)

Nowadays, one of the most important tool in risk management is the Value-at-Risk(VaR), which is a risk measure that focuses on the tail of a distribution. In this study, we are concerned with situations where the loss function can be very large which calls for the use of VaR.

**Definition 6.2.** *Given some confidence level* $\alpha \in (0, 1)$*, the VaR of the loss function* ${}_0L$ *is defined as follows:*

$$VaR_\alpha = inf\{p \in \boldsymbol{R} : P({}_0L > p) \leq 1 - \alpha\}$$

When $\alpha = 95\%$ and the distribution for ${}_0L$ is continuous, for example, then $VaR_{0.95}$ will give the 95% percentile of the loss function which means that the probability of ${}_0L$ exceeding $VaR_{0.95}$ is at most 5%. This measurement will provide us with a quantitive understanding of an "extreme loss" on one policy.

As a risk measure, VaR satisfies the non-excessive loading, translative, positive homogeneity and monotonic properties since $VaR_\alpha$ is equal to a $100\alpha\%$ percentile if the distribution of ${}_0L$ is continuous. However, the non-negative loading properties does not apply to VaR due to different choices of $\alpha$. It has been proven that VaR is not subadditive either. For example, it is possible to have $X$ and $Y$ multivariate normal with $VaR_{0.95}$ of $X+Y$ greater than the sum of $VaR_{0.95}$ for $X$ and $Y$.

As mentioned before, our focus for this study lies in the loss function of this combination hybrid pension plan. We are not only concerned about how to set up the contribution rates, we also need to investigate how to protect the plan sponsor against some unexpected loss embedded in the replacement ratio guarantee. Therefore the risk measure VaR meets our needs quite well. In practice, one of the applications of VaR is the Economic Capital(EC) in the Enterprise Risk Management (ERM) framework. Now we will introduce this concept and then calculate the EC for our combination hybrid pension plan.

## 6.3 Economic Capital

The Economic Capital is defined as the amount of risk capital required to be held by a financial service firm to remain solvent with a given probability on a going concern basis.

**Definition 6.3.** *The economic capital is defined with respect to some risk measure $\rho$ as*

$$EC[S] = \rho[S] - E[S], \tag{6.1}$$

*where $S$ is the total loss of the company. (related to some line of business)*

(See Denuit, Dhaene, Goovaerts and Kaas 2005.)

Since we are looking for the amount of risk capital required to stay solvent, some worst case scenarios need to be considered on the loss function ${}_0L$. Therefore, the risk measure VaR is commonly chosen in Equation (6.1) to represent $\rho$. In this case, $EC[S]$ is called **VaR-based Economic Capital**. It is defined as

**Definition 6.4.**

$$EC[S;\alpha] = VaR[S;\alpha] - E[S] \tag{6.2}$$

*where* $VaR[S;\alpha]$ *represent the* $VaR_\alpha$ *of variable* $S$.

For example, if $\alpha = 99.95\%$, then $EC[S;\alpha]$ represents the amount of risk capital that the insurance company needs to keep aside for unexpected losses, in order to maintain a probability of 99.95% of solvency over a certain time horizon. From here on in this study, we will simply use "Economic Capital" to refer to the VaR-based economic capital since no other risk measures are considered here.

For this study, we consider the combination hybrid pension plan as the entire business of the plan sponsor which could be an insurance company. This could be a realistic case if this pension plan is a government level or state level plan. Risks that need to be taken into account for a company usually include market risk, operational risk, credit risk and for insurance companies there is also mortality risk. This study mainly focuses on the market risk associated with a combination hybrid pension plan, therefore we will present the calculation for the market risk economic capital in this section.

In practice the economic capital is usually obtained through simulations in each risk category to calculate separate required economic capitals. Then a correlation matrix for all risk categories is applied to get an overall economic capital. The confidence level of the corresponding VaR in EC calculation of a financial service company is sometimes adopted by rating agencies in their capital adequacy assessment for that company. Therefore the concept of economic capital is not only for the insurance company itself for solvency purposes, it is also utilized by rating agencies and even regulators in some cases.

Next we will try to calculate VaR for an individual policy of this combination hybrid plan through some numerical method.

## 6.4 EC for hybrid pension plan

Now we perform simulations on the VAR(1) model studied in earlier chapters with 10 years of contributions and 10 years of pension payments, and obtain the VaR-based economic capital with several confidence levels for this combination hybrid pension plan with different investment strategies in the DC account. Once the observation data is collected from simulation, we rank these observations and look for the 99.95*th* percentile, for example. Then the economic capital is calculated by subtracting $E({}_0L|\underline{X}^0)$ from the 99.95*th* percentile.

| Confidence Level | Strategy A | Strategy B | Strategy C |
|---|---|---|---|
| 95% | 1.229 | 1.320 | 1.542 |
| 97.5% | 1.563 | 1.651 | 1.892 |
| 99% | 2.000 | 2.078 | 2.338 |
| 99.5% | 2.305 | 2.406 | 2.669 |
| 99.95% | 3.370 | 3.427 | 3.728 |

Table 6.1: VaR-based Economic Capital for all investment strategies

For each investment strategy, the replacement ratio is set to 70% and the contribution rate is set such that $E({}_0L|\underline{X}^0) = 0$. Here we also used a total number of

200,000 simulations. The results for the economic capital are shown in Table 6.1.

An obvious conclusion we can draw from Table 6.1 is that for any investment strategy, $EC[S;\alpha]$ increases in $\alpha$. That is to say, if the plan sponsor wants to be more confident that he can stay solvent, more risk capital needs to be set aside to cover large potential losses.

Secondly, the results in Table 6.1 agree quite well with Figure 5.3 and point out the significant impact of the asset allocation in the DC account on the loss function ${}_0L$. $EC[S;\alpha]$ also increases in the portion of stock in the investment strategy. In order to be confident about solvency, the plan sponsor needs to set aside more risk capital for high risk asset allocations. Please note higher economic capital would induce higher cost of capital for the plan sponsor as well since EC is only allowed to be invested in certain low risk and low return assets.

Therefore, from the economic capital point of view, the range of asset allocation in the DC account should be restricted by the plan sponsor, otherwise the plan sponsor is not only facing huge potential losses, but also inevitable costs brought by a large economic capital.

# Chapter 7

# Conclusions

In this study, we focus on the market risk of a combination hybrid pension plan which has both a DC account and a DB final salary scheme through a replacement ratio guarantee. Assumptions for a limiting portfolio are made, therefore mortality risk is not taken into account. A vector AR(1) model is applied to model the inflation, real wage increase, investment return of the DC account and long term investment return.

Using US historical data, parameter estimations for three vector AR(1) models which employ different investment strategies in the DC account are performed. Then we derive recursive formulas to calculate the first three moments of the loss function at time of issue, ${}_0L$, of a single policy given current financial market information. We find that the contribution rate required in this plan such that $E({}_0L|\underline{X}^0)$ is higher compared to regular DB and DC plans. Furthermore, with the three investment strategies considered for the DC account, both the standard deviation and skewness for the loss function ${}_0L$ are extremely high. We also try to dampen the standard deviation through adjustment of the plan feature parameters but no significant improvements are observed. Therefore, plan sponsors offering such a combination hybrid pension plan actually expose themselves to a significant amount of market risks.

The distribution of ${}_0L$ is also studied to gain more perspective about the market risk. We extend the approximation method introduced by Parker(1993a) to a two-dimensional case and obtain the distribution of ${}_0L$ for the relatively low risk investment strategy. Based on our vector AR(1) models, the accuracy of the approximation method highly depends on the number of points in the discretization, and the accuracy is higher when the number of point is increased. But the runtime of this approximation also goes up significantly with the number of points. Therefore we switch to a simulation study for the loss function and make comparisons between those two methods. The runtime issue for the simulation seems better and therefore the simulation method is adopted in further studies.

A practical application of VaR-based Economic Capital is introduced and simulation is once again used to obtain EC for all three investment strategies. Results show that in order for the plan sponsor to stay solvent with a high confidence level, a great amount of risk capital is required to be held and set aside. Also, the economic capital required increases with the portion of stock in the asset allocation of the DC account.

Therefore we conclude that the plan sponsor of a combination hybrid pension plan needs to be fully aware of the market risk that he is exposed to and price the replacement ratio guarantee very carefully. The plan sponsor should set up restrictions about how plan members can invest their contributions in the DC account, or at least charge different contribution rates for different investment strategies.

# Appendix A

# Mortality Table (CSO 2001)

| $x$ | $q_x$ | $x$ | $q_x$ | $x$ | $q_x$ | $x$ | $q_x$ | $x$ | $q_x$ | $x$ | $q_x$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0.0010 | 21 | 0.0010 | 42 | 0.0020 | 63 | 0.0137 | 84 | 0.1054 | 105 | 0.4592 |
| 1 | 0.0006 | 22 | 0.0010 | 43 | 0.0022 | 64 | 0.0152 | 85 | 0.1166 | 106 | 0.4822 |
| 2 | 0.0004 | 23 | 0.0010 | 44 | 0.0024 | 65 | 0.0169 | 86 | 0.1289 | 107 | 0.5067 |
| 3 | 0.0003 | 24 | 0.0011 | 45 | 0.0027 | 66 | 0.0185 | 87 | 0.1424 | 108 | 0.5327 |
| 4 | 0.0002 | 25 | 0.0011 | 46 | 0.0029 | 67 | 0.0201 | 88 | 0.1567 | 109 | 0.5603 |
| 5 | 0.0002 | 26 | 0.0011 | 47 | 0.0032 | 68 | 0.0219 | 89 | 0.1719 | 110 | 0.5896 |
| 6 | 0.0002 | 27 | 0.0012 | 48 | 0.0033 | 69 | 0.0236 | 90 | 0.1877 | 111 | 0.6208 |
| 7 | 0.0002 | 28 | 0.0012 | 49 | 0.0035 | 70 | 0.0258 | 91 | 0.2024 | 112 | 0.6538 |
| 8 | 0.0002 | 29 | 0.0012 | 50 | 0.0038 | 71 | 0.0282 | 92 | 0.2178 | 113 | 0.6889 |
| 9 | 0.0002 | 30 | 0.0011 | 51 | 0.0041 | 72 | 0.0313 | 93 | 0.2340 | 114 | 0.7262 |
| 10 | 0.0002 | 31 | 0.0011 | 52 | 0.0045 | 73 | 0.0346 | 94 | 0.2511 | 115 | 0.7657 |
| 11 | 0.0003 | 32 | 0.0011 | 53 | 0.0049 | 74 | 0.0381 | 95 | 0.2692 | 116 | 0.8076 |
| 12 | 0.0003 | 33 | 0.0012 | 54 | 0.0055 | 75 | 0.0419 | 96 | 0.2856 | 117 | 0.8521 |
| 13 | 0.0004 | 34 | 0.0012 | 55 | 0.0062 | 76 | 0.0461 | 97 | 0.3032 | 118 | 0.8992 |
| 14 | 0.0005 | 35 | 0.0012 | 56 | 0.0069 | 77 | 0.0509 | 98 | 0.3219 | 119 | 0.9492 |
| 15 | 0.0006 | 36 | 0.0013 | 57 | 0.0076 | 78 | 0.0566 | 99 | 0.3419 | 120 | 1.0000 |
| 16 | 0.0007 | 37 | 0.0013 | 58 | 0.0083 | 79 | 0.0631 | 100 | 0.3632 | | |
| 17 | 0.0009 | 38 | 0.0014 | 59 | 0.0090 | 80 | 0.0701 | 101 | 0.3801 | | |
| 18 | 0.0009 | 39 | 0.0015 | 60 | 0.0099 | 81 | 0.0782 | 102 | 0.3981 | | |
| 19 | 0.0010 | 40 | 0.0017 | 61 | 0.0109 | 82 | 0.0865 | 103 | 0.4172 | | |
| 20 | 0.0010 | 41 | 0.0018 | 62 | 0.0123 | 83 | 0.0955 | 104 | 0.4376 | | |

(Data source: http://www.actuary.org/life/CSO_0702.asp)

# Appendix B

# Data for Model Estimation

- Inflation (percentage)
  (Data source: http://www.miseryindex.us/)

| Time | Data | Time | Data | Time | Data | Time | Data | Time | Data |
|---|---|---|---|---|---|---|---|---|---|
| 87-03 | 3.03 | 91-06 | 4.70 | 95-09 | 2.54 | 99-12 | 2.68 | 04-03 | 1.74 |
| 87-06 | 3.65 | 91-09 | 3.39 | 95-12 | 2.54 | 00-03 | 3.76 | 04-06 | 3.27 |
| 87-09 | 4.36 | 91-12 | 3.06 | 96-03 | 2.84 | 00-06 | 3.73 | 04-09 | 2.54 |
| 87-12 | 4.43 | 92-03 | 3.19 | 96-06 | 2.75 | 00-09 | 3.45 | 04-12 | 3.26 |
| 88-03 | 3.93 | 92-06 | 3.09 | 96-09 | 3.00 | 00-12 | 3.39 | 05-03 | 3.15 |
| 88-06 | 3.96 | 92-09 | 2.99 | 96-12 | 3.32 | 01-03 | 2.92 | 05-06 | 2.53 |
| 88-09 | 4.17 | 92-12 | 2.90 | 97-03 | 2.76 | 01-06 | 3.25 | 05-09 | 4.69 |
| 88-12 | 4.42 | 93-03 | 3.09 | 97-06 | 2.30 | 01-09 | 2.65 | 05-12 | 3.42 |
| 89-03 | 4.98 | 93-06 | 3.00 | 97-09 | 2.15 | 01-12 | 1.55 | 06-03 | 3.36 |
| 89-06 | 5.17 | 93-09 | 2.69 | 97-12 | 1.70 | 02-03 | 1.48 | 06-06 | 4.32 |
| 89-09 | 4.34 | 93-12 | 2.75 | 98-03 | 1.37 | 02-06 | 1.07 | 06-09 | 2.06 |
| 89-12 | 4.65 | 94-03 | 2.51 | 98-06 | 1.68 | 02-09 | 1.51 | 06-12 | 2.54 |
| 90-03 | 5.23 | 94-06 | 2.49 | 98-09 | 1.49 | 02-12 | 2.38 | 07-03 | 2.78 |
| 90-06 | 4.67 | 94-09 | 2.96 | 98-12 | 1.61 | 03-03 | 3.02 | 07-06 | 2.69 |
| 90-09 | 6.16 | 94-12 | 2.67 | 99-03 | 1.73 | 03-06 | 2.11 | 07-09 | 2.76 |
| 90-12 | 6.11 | 95-03 | 2.85 | 99-06 | 1.96 | 03-09 | 2.32 | 07-12 | 4.08 |
| 91-03 | 4.90 | 95-06 | 3.04 | 99-09 | 2.63 | 03-12 | 1.88 | | |

- Personal Annual Income Data (in USD)
(Data source: http://www.bea.gov/)

| Time | Salary | Time | Salary | Time | Salary |
|---|---|---|---|---|---|
| 1987-I | 15,897 | 1993-IV | 21,766 | 2000-III | 30,106 |
| 1987-II | 16,084 | 1994-I | 21,616 | 2000-IV | 30,205 |
| 1987-III | 16,342 | 1994-II | 22,104 | 2001-I | 30,565 |
| 1987-IV | 16,694 | 1994-III | 22,325 | 2001-II | 30,599 |
| 1988-I | 16,951 | 1994-IV | 22,656 | 2001-III | 30,563 |
| 1988-II | 17,196 | 1995-I | 22,881 | 2001-IV | 30,558 |
| 1988-III | 17,496 | 1995-II | 22,992 | 2002-I | 30,697 |
| 1988-IV | 17,785 | 1995-III | 23,126 | 2002-II | 30,894 |
| 1989-I | 18,284 | 1995-IV | 23,311 | 2002-III | 30,825 |
| 1989-II | 18,454 | 1996-I | 23,730 | 2002-IV | 30,852 |
| 1989-III | 18,606 | 1996-II | 24,106 | 2003-I | 31,037 |
| 1989-IV | 18,833 | 1996-III | 24,307 | 2003-II | 31,355 |
| 1990-I | 19,197 | 1996-IV | 24,557 | 2003-III | 31,593 |
| 1990-II | 19,466 | 1997-I | 24,928 | 2003-IV | 31,988 |
| 1990-III | 19,651 | 1997-II | 25,128 | 2004-I | 32,408 |
| 1990-IV | 19,684 | 1997-III | 25,446 | 2004-II | 32,837 |
| 1991-I | 19,685 | 1997-IV | 25,830 | 2004-III | 33,235 |
| 1991-II | 19,859 | 1998-I | 26,358 | 2004-IV | 34,017 |
| 1991-III | 19,970 | 1998-II | 26,753 | 2005-I | 34,110 |
| 1991-IV | 20,173 | 1998-III | 27,067 | 2005-II | 34,578 |
| 1992-I | 20,494 | 1998-IV | 27,336 | 2005-III | 34,809 |
| 1992-II | 20,750 | 1999-I | 27,538 | 2005-IV | 35,525 |
| 1992-III | 20,912 | 1999-II | 27,715 | 2006-I | 36,186 |
| 1992-IV | 21,320 | 1999-III | 27,967 | 2006-II | 36,534 |
| 1993-I | 20,922 | 1999-IV | 28,507 | 2006-III | 36,823 |
| 1993-II | 21,331 | 2000-I | 29,385 | 2006-IV | 37,290 |
| 1993-III | 21,401 | 2000-II | 29,687 | | |

- S&P 500 Index
  (Data source: http://finance.yahoo.com/)

| Time | Adj. Close | Time | Adj. Close | Time | Adj. Close |
|---|---|---|---|---|---|
| Feb-87 | 284 | Nov-93 | 462 | Aug-00 | 1,518 |
| May-87 | 290 | Feb-94 | 467 | Nov-00 | 1,315 |
| Aug-87 | 330 | May-94 | 457 | Feb-01 | 1,240 |
| Nov-87 | 230 | Aug-94 | 475 | May-01 | 1,256 |
| Feb-88 | 268 | Nov-94 | 454 | Aug-01 | 1,134 |
| May-88 | 262 | Feb-95 | 487 | Nov-01 | 1,139 |
| Aug-88 | 262 | May-95 | 533 | Feb-02 | 1,107 |
| Nov-88 | 274 | Aug-95 | 562 | May-02 | 1,067 |
| Feb-89 | 289 | Nov-95 | 605 | Aug-02 | 916 |
| May-89 | 321 | Feb-96 | 640 | Nov-02 | 936 |
| Aug-89 | 351 | May-96 | 669 | Feb-03 | 841 |
| Nov-89 | 346 | Aug-96 | 652 | May-03 | 964 |
| Feb-90 | 332 | Nov-96 | 757 | Aug-03 | 1,008 |
| May-90 | 361 | Feb-97 | 791 | Nov-03 | 1,058 |
| Aug-90 | 323 | May-97 | 848 | Feb-04 | 1,145 |
| Nov-90 | 322 | Aug-97 | 899 | May-04 | 1,121 |
| Feb-91 | 367 | Nov-97 | 955 | Aug-04 | 1,104 |
| May-91 | 390 | Feb-98 | 1,049 | Nov-04 | 1,174 |
| Aug-91 | 395 | May-98 | 1,091 | Feb-05 | 1,204 |
| Nov-91 | 375 | Aug-98 | 957 | May-05 | 1,192 |
| Feb-92 | 413 | Nov-98 | 1,164 | Aug-05 | 1,220 |
| May-92 | 415 | Feb-99 | 1,238 | Nov-05 | 1,249 |
| Aug-92 | 414 | May-99 | 1,302 | Feb-06 | 1,281 |
| Nov-92 | 431 | Aug-99 | 1,320 | May-06 | 1,270 |
| Feb-93 | 443 | Nov-99 | 1,389 | Aug-06 | 1,304 |
| May-93 | 450 | Feb-00 | 1,366 | Nov-06 | 1,401 |
| Aug-93 | 464 | May-00 | 1,421 | | |

- 1-year treasury bond return (percentage)
  (Data source: http://www.federalreserve.gov/)

| Time | Return | Time | Return | Time | Return |
|---|---|---|---|---|---|
| Mar-87 | 6.03 | Jun-94 | 5.27 | Sep-01 | 2.82 |
| Jun-87 | 6.80 | Sep-94 | 5.76 | Dec-01 | 2.22 |
| Sep-87 | 7.67 | Dec-94 | 7.14 | Mar-02 | 2.57 |
| Dec-87 | 7.17 | Mar-95 | 6.43 | Jun-02 | 2.20 |
| Mar-88 | 6.71 | Jun-95 | 5.64 | Sep-02 | 1.72 |
| Jun-88 | 7.49 | Sep-95 | 5.62 | Dec-02 | 1.45 |
| Sep-88 | 8.09 | Dec-95 | 5.31 | Mar-03 | 1.24 |
| Dec-88 | 8.99 | Mar-96 | 5.34 | Jun-03 | 1.01 |
| Mar-89 | 9.57 | Jun-96 | 5.81 | Sep-03 | 1.24 |
| Jun-89 | 8.44 | Sep-96 | 5.83 | Dec-03 | 1.31 |
| Sep-89 | 8.22 | Dec-96 | 5.47 | Mar-04 | 1.19 |
| Dec-89 | 7.72 | Mar-97 | 5.80 | Jun-04 | 2.12 |
| Mar-90 | 8.35 | Jun-97 | 5.69 | Sep-04 | 2.12 |
| Jun-90 | 8.10 | Sep-97 | 5.52 | Dec-04 | 2.67 |
| Sep-90 | 7.76 | Dec-97 | 5.53 | Mar-05 | 3.30 |
| Dec-90 | 7.05 | Mar-98 | 5.39 | Jun-05 | 3.36 |
| Mar-91 | 6.40 | Jun-98 | 5.41 | Sep-05 | 3.85 |
| Jun-91 | 6.36 | Sep-98 | 4.71 | Dec-05 | 4.35 |
| Sep-91 | 5.57 | Dec-98 | 4.52 | Mar-06 | 4.77 |
| Dec-91 | 4.38 | Mar-99 | 4.78 | Jun-06 | 5.16 |
| Mar-92 | 4.63 | Jun-99 | 5.10 | Sep-06 | 4.97 |
| Jun-92 | 4.17 | Sep-99 | 5.25 | Dec-06 | 4.94 |
| Sep-92 | 3.18 | Dec-99 | 5.84 | Mar-07 | 4.92 |
| Dec-92 | 3.71 | Mar-00 | 6.22 | Jun-07 | 4.96 |
| Mar-93 | 3.33 | Jun-00 | 6.17 | Sep-07 | 4.14 |
| Jun-93 | 3.54 | Sep-00 | 6.13 | Dec-07 | 3.26 |
| Sep-93 | 3.36 | Dec-00 | 5.60 | Mar-08 | 1.54 |
| Dec-93 | 3.61 | Mar-01 | 4.30 | | |
| Mar-94 | 4.32 | Jun-01 | 3.58 | | |

- 10-year treasury bond return (percentage)
  (Data source: http://www.federalreserve.gov/)

| Time | Return | Time | Return | Time | Return |
|---|---|---|---|---|---|
| Mar-87 | 7.25 | Jun-94 | 7.10 | Sep-01 | 4.73 |
| Jun-87 | 8.40 | Sep-94 | 7.46 | Dec-01 | 5.09 |
| Sep-87 | 9.42 | Dec-94 | 7.81 | Mar-02 | 5.28 |
| Dec-87 | 8.99 | Mar-95 | 7.20 | Jun-02 | 4.93 |
| Mar-88 | 8.37 | Jun-95 | 6.17 | Sep-02 | 3.87 |
| Jun-88 | 8.92 | Sep-95 | 6.20 | Dec-02 | 4.03 |
| Sep-88 | 8.98 | Dec-95 | 5.71 | Mar-03 | 3.81 |
| Dec-88 | 9.11 | Mar-96 | 6.27 | Jun-03 | 3.33 |
| Mar-89 | 9.36 | Jun-96 | 6.91 | Sep-03 | 4.27 |
| Jun-89 | 8.28 | Sep-96 | 6.83 | Dec-03 | 4.27 |
| Sep-89 | 8.19 | Dec-96 | 6.30 | Mar-04 | 3.83 |
| Dec-89 | 7.84 | Mar-97 | 6.69 | Jun-04 | 4.73 |
| Mar-90 | 8.59 | Jun-97 | 6.49 | Sep-04 | 4.13 |
| Jun-90 | 8.48 | Sep-97 | 6.21 | Dec-04 | 4.23 |
| Sep-90 | 8.89 | Dec-97 | 5.81 | Mar-05 | 4.50 |
| Dec-90 | 8.08 | Mar-98 | 5.65 | Jun-05 | 4.00 |
| Mar-91 | 8.11 | Jun-98 | 5.50 | Sep-05 | 4.20 |
| Jun-91 | 8.28 | Sep-98 | 4.81 | Dec-05 | 4.47 |
| Sep-91 | 7.65 | Dec-98 | 4.65 | Mar-06 | 4.72 |
| Dec-91 | 7.09 | Mar-99 | 5.23 | Jun-06 | 5.11 |
| Mar-92 | 7.54 | Jun-99 | 5.90 | Sep-06 | 4.72 |
| Jun-92 | 7.26 | Sep-99 | 5.92 | Dec-06 | 4.56 |
| Sep-92 | 6.42 | Dec-99 | 6.28 | Mar-07 | 4.56 |
| Dec-92 | 6.77 | Mar-00 | 6.26 | Jun-07 | 5.10 |
| Mar-93 | 5.98 | Jun-00 | 6.10 | Sep-07 | 4.52 |
| Jun-93 | 5.96 | Sep-00 | 5.80 | Dec-07 | 4.10 |
| Sep-93 | 5.36 | Dec-00 | 5.24 | Mar-08 | 3.51 |
| Dec-93 | 5.77 | Mar-01 | 4.89 | | |
| Mar-94 | 6.48 | Jun-01 | 5.28 | | |

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