RUIN PROBLEM IN RETIREMENT UNDER STOCHASTIC RETURN RATE AND MORTALITY RATE AND ITS APPLICATIONS

by

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PROJECT SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF SCIENCE

In the
Department of Statistics and Actuarial Science

SIMON FRASER UNIVERSITY

Spring 2008

APPROVAL

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Degree: **Master of Science**

Title of Project: **Ruin problem in retirement under stochastic return rate and mortality rate and its applications.**

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ABSTRACT

Retirees face a difficult choice between annuitization from insurance firms and self-management or so-called self-annuitization. Self-annuitization could provide a higher consumption by investing more assets on equity market but with a risk that retirees may outlive the income from their self-managed assets. Using the Ornstein-Uhlenbeck stochastic model, also called the Vasicek model, for the rate of return, we focus our study on the ruin probability in retirement. We show how asset mix, initial rate of return, and gender impact the ruin probability in retirement. We derive a recursive formula to calculate an approximate distribution for the present value of the life annuity function under our stochastic model. Finally, we use our model to illustrate how a VaR technique can help determine the optimal consumption for a retiree with a certain tolerance to ruin under different retirement goals.

Keywords: Ruin, Stochastic Interest Model, Life Annuity, Optimal Consumption, Approximate Distribution, VaR, Ornstein-Uhlenbeck (O-U) Model, Vasicek Model

DEDICATION

This project is dedicated to my lovely newborn baby girl Ruby and her energetic, three and half years old, “Big” brother Alexander who is so excited with his new little sister.

ACKNOWLEDGEMENTS

I would like to thank all of those people who have helped me go through the chapter of my student life at SFU.

First, I would like to express my deepest gratitude to my supervisor, Professor Gary Parker, who guided me into the actuarial career, spent considerate time to mentor me through my graduate studies, and gave me valuable advice on this project. I am thankful to him to give me the flexibility to complete my graduate program while raising a family. He is a role model of dedication to students and academia that will be one of the greatest assets I learnt from this program.

I would like to thank all of the faculty and staffs in this department. I am very grateful to my examining committee, Dr. Yi Lu and Dr. Tim Swartz, for their very careful reading of the project report and valuable criticisms that helped me perfect its final version. Also thanks to Dr. Cary Tsai and Barbara Sanders, for their courses which helped me acquire the knowledge and skills necessary for my future actuarial career.

This journey would have been lonely and boring without the input and help from my fellow students. Thanks to Suli Ma, Lihui Zhao, Luyao Lin, Donghong Wu, YunFeng Dai, Wei Qian, Yinan Jiang, Monica Lu. Special thanks to Feng

Gao, who was a very patient listener when I wanted someone to confirm the ideas in my project and willing to give me serious feedbacks. Many thanks to Huanhuan Wu, who helped me find a place to stay on campus and provided dinners with her family during my defence days.

Last but not least, I owe a great debt of gratitude to my wife for her love, support, understanding, patience, and complete dedication; for giving birth to our beautiful baby girl during this challenging time; for raising the two children on her own most of the time without complaining too much.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1: INTRODUCTION

During early retirement, many retirees face a difficult choice between annuitization from insurance firms and discretionary management of assets with systematic withdrawals for consumption purpose or so-called “self-annuitization”. For example, in Canada, the government requires retirees to convert their RRSPs to one or more retirement income sources[1] by December 31st of the year they reach age 71. One option for retirees is to use these funds to purchase a life annuity from insurance firms. The other is to transfer them to a life income fund(LIF) or life retirement income fund(LRIF) for which retirees will self-manage their funds while required to make an annual minimum withdrawal based on age.

Annuitization will assure a lifelong consumption stream that cannot be outlived. However, it may be quite expensive and come at the cost of a complete loss of liquidity. Self-annuitization, on the other hand, will have the flexibility of allowing the accumulation of wealth during retirement through potentially higher returns from the market. In addition, it could leave a substantial bequest to survivors and estates at the death of retirees. However, this comes with a risk that retirees will not be able to maintain their planned standard of living during retirement.

[1] In this project, for simplicity we only focus on registered retirement funds for which annuitization and self-annuitization have the same tax treatment.

"Ruin probability in retirement" is the probability of running out of self-managed asset under self-annuitization. Its study could help retirees make an informed decision based on their own view of risk, desired standard of living for retirement, economic and social environment. The distribution of the present value of a whole life annuity under stochastic interest/return rate and mortality is the focus in this study, but with different models and approaches. In our project, we assume an Ornstein-Uhlenbeck (O-U) model[2] for the rate of return[3] and study the distribution of the present value of a whole life annuity function and the ruin probability. Furthermore, we use the results to consider some practical applications.

The remainder of this project is organized as follows. Chapter 2 gives a review on the stochastic interest models and the ruin problem in retirement in published actuarial literature. In Chapter 3, we define the functions of the rate of return, the present value, the life annuity and derive their moments under the O-U rate of return model. In order to compute the distribution of the present value of the whole life annuity function, three approaches are presented and compared in Chapter 4. In Chapter 5, we study the ruin probability under the O-U rate of return model and analyze the impact from the asset allocation strategy, initial interest rate and gender. Chapter 6 discusses some practical applications in retirement with the consideration of ruin probability that could help retirees make

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[2] It is often called the Vasicek model of interest rates in finance.

[3] In the project, the rate of return is defined as the instantaneous rate of return, also known as the force of interest in the actuarial literature.

their decision about self-annuitization. Finally, Chapter 7 gives a brief conclusion and proposes further studies.

CHAPTER 2: REVIEW OF STOCHASTIC INTEREST MODELS AND RUIN PROBLEM IN RETIREMENT

This chapter gives a review of stochastic interest models and the ruin problem in retirement studied in the actuarial literature.

2.1 Stochastic Interest Models in the Actuarial Literature

The topic of random interest rates has been intensively studied in actuarial and finance literatures over the last few decades. The following publications are some of the papers which study actuarial functions under the context of random interest rates.

Boyle (1976) was one of the first to analyze the statistical properties of an annuity and insurance contract under stochastic returns. By assuming that the returns of investment per year are independently and identically distributed, he derived the first three moments of interest discount functions and further applied them to traditional actuarial life contingency functions.

Panjer and Bellhouse (1980) continued on this theme by generalizing Boyle's work. They proposed to use first and second order autoregressive models for the force of interest and gave a general method to derive the first two moments of life insurance and annuity functions. The stationary process for the autoregressive model was used in this paper while conditional models were developed and applied to interest, insurance and annuity functions in Bellhouse and Panjer (1981).

Waters(1978) presented a general method of finding moments of insurance and annuity functions under the assumption that force of interest rates are independently, identically and normally distributed. He also studied the moments of portfolios of policies and fitted the distribution for an infinite size portfolio with a Pearson curve to obtain the percentiles.

Beekman and Fuelling (1990) derived the first two moments of life annuity function by modelling the force of interest accumulation function with an O-U process. Later Beekman and Fuelling (1991) introduced a Wiener process to model the force of interest accumulation function which could generate more variations for life annuity functions.

Parker (1992, 1993a, 1994a, 1994b) used approximation techniques to derive a recursive formula for the cumulative distribution function (CDF) of the present value of a portfolio of insurances or annuities. Parker (1993b) compared the modelling of the force of interest and the force of interest accumulation function for three stochastic interest models - White Noise process, Wiener process, O-U process. He argued that modelling the force of interest has some advantages by looking at a particular conditional expectation of the force of interest accumulation function.

2.2 Ruin Problem in Retirement

We now review the ruin problem in retirement. Dufresne (1990) derived the probability density function of the present value of a perpetuity subjected to a stochastic Wiener rate of interest and proved that it is inverse gamma distributed.

Milevsky et al. (1997) used simulations to study the ruin probability in a model with the stochastic investment return and mortality. They found that retirees could reduce the probability of ruin by investing part of their assets in a higher return and higher risk asset like the common equity. They also found that the probability of ruin is much higher for a female than for a male of the same age with the same wealth to consumption ratio. They studied the bequest under alternative asset allocation strategies and found that females tend to have lower value of bequest than males at lower equity allocation while they tend to have higher value of bequest at higher equity allocation.

Milevsky (1998) developed a stochastic model in which retirees defer annuitization until it is no longer possible to beat the mortality-adjusted rate of return from a life annuity. Their model incorporated three stochastic processes: the return from asset, the internal rate of return for a life annuity, and the mortality rate. They concluded that under the current environment, a female (male) at age 65 has 90% (85%) chance of beating the rate of return from a life annuity until age 80.

Ruin probability in retirement is equal to the probability that the stochastic present value of future consumption is greater than the initial wealth. Milevsky and Robinson (2000) studied the approximate distribution of a whole life annuity function. They used Gompertz law to model mortality and a geometric Brownian motion to model asset price. They fitted the stochastic present value of continuous whole life annuity with the reciprocal gamma and Type II Johnson distributions and validated these two approximations with the result from simulations. A numerical case was illustrated to show the impact on the ruin probability from asset allocation strategy and gender. In their example, a well-diversified portfolio (80% equity / 20% long-term bond for male and 60% equity / 40% long-term bond for female) will achieve the lowest ruin probability. Under the same asset allocation strategy, females will have a higher ruin probability than males due to the longevity.

Blake et al. (2003) studied the strategy to postpone annuitization. They used simulations to compare the purchase of a conventional life annuity at age 65 with two other distribution programmes: equity-linked annuity (ELA) with a level life annuity purchased at age 75 and equity-linked income-drawdown (ELID) with a level life annuity purchased at age 75. ELA and ELID are very similar except for the fact that the plan member in ELID will receive a survival credit at the start of each year if he survives before age 75 but has to surrender his bequest to the life office if he dies before age 75. They found that the most important decision, in terms of the cost to the plan member, is the level of equity investment. They also found that the optimal age to annuitize depends on the bequest utility and the investment performance of the fund during retirement.

Huang et al. (2004) implemented numerical PDE (partial differential equation) solution techniques to compute the ruin probability in retirement. They compared their PDE-based values with those quick-and-dirty heuristic approximation methods widely used for ruin problem, such as the reciprocal gamma approximation (RG), the lognormal approximation (LN), and the comonotonic-based lower bound approximation (CLB). One of their conclusions is that the RG approximation proposed by Milevksy and Robinson (2000) will break down at a high level of volatility. The CLB proposed by Dhaene et al. (2002a, 2002b) is better than the other two approximations when the time horizon is fixed. However, the CLB approximation needs to fix the time horizon which makes it non-applicable in the study of the ruin problem under a stochastic mortality rate.

CHAPTER 3: FUNCTIONS RELATED TO THE O-U RATE OF RETURN MODEL

In this chapter, we define the functions related to the rate of investment return in the context of life contingency and derive their moments under the O-U model for the rate of return. The later chapters will use this model to study the ruin problem in retirement.

3.1 O-U Model for Rate of Return

3.1.1 Rate of Return and Moments

Under the O-U model, the (instantaneous) rate of (investment) return (also known as the force of interest) δ_t can be defined as (see Parker (1992))

$$d\delta_t = -\alpha(\delta_t - \delta)dt + \sigma dW_t \ , \qquad (2.1)$$

where δ is the long term mean of the rate of return, α is the friction force bringing the process δ_t back towards its long-term mean, σ is the so-called diffusion coefficient, and W_t is a standard Brownian Motion. We can solve the differential equation (2.1) as

$$\delta_t = \delta + e^{-\alpha t}(\delta_0 - \delta) + \int_0^t e^{-\alpha(t-s)}\sigma \, dW_s \ , \qquad (2.2)$$

with initial value δ_0 which is the rate of return at time 0.

Then δ_t is a Gaussian process with mean

$$E(\delta_t) = e^{-\alpha t}(\delta_0 - \delta) + \delta \tag{2.3}$$

and variance

$$Var(\delta_t) = \frac{\sigma^2}{2\alpha}(1 - e^{-2\alpha t}) \ . \tag{2.4}$$

For s ≤ t, the autocovariance function is

$$COV(\delta_t, \delta_s) = e^{-\alpha(t+s)} \frac{\sigma^2}{2\alpha}(e^{2\alpha s} - 1) \ . \tag{2.5}$$

3.1.2 Rate of Return Accumulation Function and Moments

Define the rate of return accumulation function as

$$Y(t) = \int_0^t \delta_r \ dr \ . \tag{2.6}$$

$Y(t)$ is a Gaussian process with mean

$$E(Y(t)) = (\delta_0 - \delta)\frac{1 - e^{-\alpha t}}{\alpha} + \delta t \tag{2.7}$$

and variance

$$Var(Y(t)) = \frac{\sigma^2}{\alpha^2} t + \frac{\sigma^2}{2\alpha^3}(-3 + 4e^{-\alpha t} - e^{-2\alpha t}) \ . \tag{2.8}$$

For s ≤ t, the autocovariance function is

$$COV(Y(s), Y(t)) = \frac{\sigma^2}{\alpha^2} s + \frac{\sigma^2}{2\alpha^3}(-2 + 2e^{-\alpha s} + 2e^{-\alpha t} - e^{-\alpha(t-s)} - e^{-\alpha(t+s)}) \ . \tag{2.9}$$

3.1.3 Present Value Function and Moments

Since Y(t) is normally distributed, the present value function, $e^{-Y(t)}$, is lognormally distributed. We can derive the moments of the present value function, $e^{-Y(t)}$, by using the moment generating function of a normal distribution. We then have

$$E(e^{-Y(t)}) = \exp(-E(Y(t)) + .5 * Var(Y(t))) , \tag{2.10}$$

$$\begin{aligned} E(e^{-(Y(t)+Y(s))}) = \exp(-(E(Y(t)) + E(Y(s))) + .5*(Var(Y(t)) + Var(Y(s)) \\ + 2*COV(Y(s), Y(t)))) , \end{aligned} \tag{2.11}$$

$$\begin{aligned} Var(e^{-Y(t)}) &= E(e^{-2Y(t)}) - E^2(e^{-Y(t)}) \\ &= \exp(-2E(Y(t)) + 2*Var(Y(t))) - \exp(-2E(Y(t)) + Var(Y(t))) \\ &= \exp(-2E(Y(t)) + Var(Y(t)))(\exp(Var(Y(t))) - 1) . \end{aligned} \tag{2.12}$$

3.1.4 Present Value of Whole Life Annuity Function and Moments

First, we define the present value of the continuous whole life annuity function as

$$\tilde{a}_{\overline{T|}} = \int_0^T e^{-Y(t)} dt \ , \tag{2.13}$$

where T is the future lifetime of a person age x. The moments of the present value of a continuous whole life annuity are:

$$E(\tilde{a}_{\overline{T|}}) = E_Y(E(\tilde{a}_{\overline{T|}} \mid \{Y(t)\}) = E_Y(\int_0^\infty e^{-Y(t)} {}_t p_x \, dt) = \int_0^\infty E(e^{-Y(t)}) {}_t p_x \, dt \ , \tag{2.14}$$

$$E(\tilde{a}_{\overline{T}|}^{\,2}) = E_Y(E(\tilde{a}_{\overline{T}|}^{\,2} \mid \{Y(t)\})) = E_Y(\int_0^\infty 2\int_0^t e^{-Y(s)} ds\, e^{-Y(t)}\, {}_t p_x\, dt)$$
$$= 2\int_0^\infty \int_0^t E(e^{-(Y(s)+Y(t))})\, {}_t p_x\, ds dt\,. \qquad (2.15)$$

In our project, we study the present value of a discrete whole life annuity defined as

$$\ddot{a}_{\overline{K+1}|} = \sum_{t=0}^{K} e^{-Y(t)}\ , \qquad (2.16)$$

where K is the curtate future lifetime for a retiree at age x and we assume the limiting age is ω. The first moment for the present value of a discrete whole life annuity function is derived as:

$$E(\ddot{a}_{\overline{K+1}|}) = E_Y(E(\ddot{a}_{\overline{K+1}|} \mid \{Y(t)\})) = E_Y(\sum_{t=0}^{\omega-x-1} e^{-Y(t)}\, {}_t p_x) = \sum_{t=0}^{\omega-x-1} E(e^{-Y(t)})\, {}_t p_x\ . \qquad (2.17)$$

For the second moment, $E(\ddot{a}_{\overline{K+1}|}^{\,2}) = E_Y(E(\ddot{a}_{\overline{K+1}|}^{\,2} \mid \{Y(t)\}))$. Since

$$E(\ddot{a}_{\overline{K+1}|}^{\,2} \mid \{Y(t)\}) = \sum_{k=0}^{\omega-x-1}(\sum_{t=0}^{k} e^{-Y(t)})^2 ({}_k p_x - {}_{k+1} p_x) = \sum_{k=0}^{\omega-x-1}(\sum_{t=0}^{k}\sum_{r=0}^{k} e^{-Y(t)-Y(r)})({}_k p_x - {}_{k+1} p_x)$$
$$= \sum_{k=0}^{\omega-x-1}(\sum_{t=0}^{k}\sum_{r=0}^{t-1} 2e^{-Y(t)-Y(r)} + \sum_{t=0}^{k} e^{-2Y(t)})({}_k p_x - {}_{k+1} p_x)$$
$$= \sum_{t=0}^{\omega-x-1}(\sum_{r=0}^{t-1}\sum_{k=t}^{\omega-x-1} 2e^{-Y(t)-Y(r)} + \sum_{k=t}^{\omega-x-1} e^{-2Y(t)})({}_k p_x - {}_{k+1} p_x)$$
$$= 2\sum_{t=0}^{\omega-x-1}\sum_{r=0}^{t-1} e^{-Y(t)-Y(r)}\ {}_t p_x + \sum_{t=0}^{\omega-x-1} e^{-2Y(t)}\ {}_t p_x\ ,$$

we then have

$$E(\ddot{a}_{\overline{K+1}|}^{\,2}) = 2\sum_{t=0}^{\omega-x-1}\sum_{r=0}^{t-1} E(e^{-Y(t)-Y(r)})\ {}_t p_x + \sum_{t=0}^{\omega-x-1} E(e^{-2Y(t)})\, {}_t p_x\ \ . \qquad (2.18)$$

Similarly, we can derive the third moment. From

$$E(\ddot{a}_{\overline{K+1|}}{}^3 \mid \{Y(t)\}) = \sum_{k=0}^{\omega-x-1} (\sum_{t=0}^{k} e^{-Y(t)})^3 ({}_k p_x - {}_{k+1}p_x) = \sum_{k=0}^{\omega-x-1} (\sum_{t_1=0}^{k}\sum_{t_2=0}^{k}\sum_{t_3=0}^{k} e^{-Y(t_1)-Y(t_2)-Y(t_3)})({}_k p_x - {}_{k+1}p_x)$$

$$= \sum_{t=0}^{\omega-x-1} e^{-3Y(t)}\, {}_t p_x + 3\sum_{t_1=0}^{\omega-x-1}\sum_{t_2=0}^{t_1-1} (e^{-2Y(t_1)-Y(t_2)} + e^{-2Y(t_2)-Y(t_1)})\, {}_{t_1} p_x + 6\sum_{t_1=0}^{\omega-x-1}\sum_{t_2=0}^{t_1-1}\sum_{t_3=0}^{t_2-1} e^{-Y(t_1)-Y(t_2)-Y(t_3)}\, {}_{t_1} p_x ,$$

we have

$$E(\ddot{a}_{\overline{K+1|}}{}^3) = \sum_{k=0}^{\omega-x-1}\sum_{t_1=0}^{k}\sum_{t_2=0}^{k}\sum_{t_3=0}^{k} E(e^{-Y(t_1)-Y(t_2)-Y(t_3)})({}_k p_x - {}_{k+1}p_x)$$

$$= \sum_{t=0}^{\omega-x-1} E(e^{-3Y(t)})\, {}_t p_x + 3\sum_{t_1=0}^{\omega-x-1}\sum_{t_2=0}^{t_1-1} (E(e^{-2Y(t_1)-Y(t_2)}) + E(e^{-2Y(t_2)-Y(t_1)}))\, {}_{t_1} p_x \qquad (2.19)$$

$$+ 6\sum_{t_1=0}^{\omega-x-1}\sum_{t_2=0}^{t_1-1}\sum_{t_3=0}^{t_2-1} E(e^{-Y(t_1)-Y(t_2)-Y(t_3)})\, {}_{t_1} p_x .$$

The fourth moment is derived as:

$$E(\ddot{a}_{\overline{K+1|}}{}^4) = \sum_{k=0}^{\omega-x-1}\sum_{t_1=0}^{k}\sum_{t_2=0}^{k}\sum_{t_3=0}^{k}\sum_{t_4=0}^{k} E(e^{-Y(t_1)-Y(t_2)-Y(t_3)-Y(t_4)})({}_k p_x - {}_{k+1}p_x)$$

$$= \sum_{t=0}^{\omega-x-1} E(e^{-4Y(t)})\, {}_t p_x$$

$$+ \sum_{t_1=0}^{\omega-x-1}\sum_{t_2=0}^{t_1-1} (4E(e^{-Y(t_1)-3Y(t_2)}) + 4E(e^{-3Y(t_1)-Y(t_2)}) + 6E(e^{-2Y(t_1)-2Y(t_2)}))\, {}_{t_1} p_x \qquad (2.20)$$

$$+12\sum_{t_1=0}^{\omega-x-1}\sum_{t_2=0}^{t_1-1}\sum_{t_3=0}^{t_2-1} (E(e^{-2Y(t_1)-Y(t_2)-Y(t_3)}) + E(e^{-Y(t_1)-2Y(t_2)-Y(t_3)}) + E(e^{-Y(t_1)-Y(t_2)-2Y(t_3)}))\, {}_{t_1} p_x$$

$$+24\sum_{t_1=0}^{\omega-x-1}\sum_{t_2=0}^{t_1-1}\sum_{t_3=0}^{t_2-1}\sum_{t_4=0}^{t_3-1} E(e^{-Y(t_1)-Y(t_2)-Y(t_3)-Y(t_4)})\, {}_{t_1} p_x .$$

Later, we will use the first four moments of the present value of the whole life annuity function given by (2.17), (2.18), (2.19) and (2.20) to validate the distributions obtained from an approximate method and simulations.

3.2 Parameters and Assumptions

3.2.1 Parameter Sets for the O-U Model for the Rate of Return

In order to study the ruin probability using the O-U model for the rate of return, we need to choose some parameter sets for the model that include as many real-economic scenarios as possible. Furthermore, although the parameters are used for illustration purposes, we hope that the conclusions drawn from these parameter sets will provide practical answers to the applications presented in later chapters.

We estimate the parameter sets from the actual financial market data of the past 20 years. The rates of return on S&P 500, 10-year Constant Maturity Treasure rates and 1-year Constant Maturity Treasure rates are the proxies for rates of return on equity, long-term bond and short-term T-bill respectively.

Figure 3-1 shows the rates of return from April of 1986 to March of 2006 in real term. Note that the rate of return in real term, which is adjusted for inflation, will be used throughout the whole project since we model the retirees' consumption based on real term.

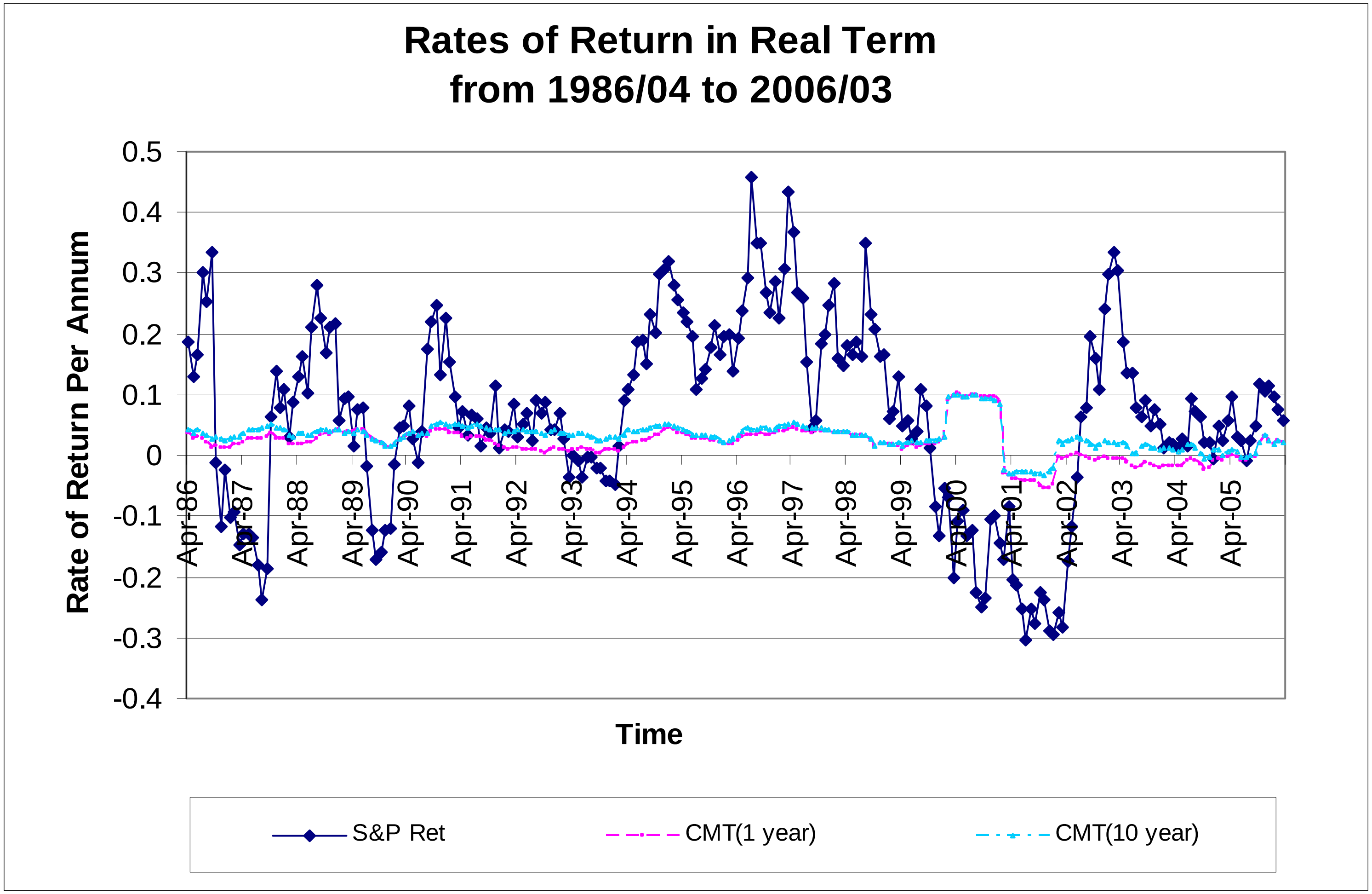


Figure 3-1 Rates of Return in Real Term (Adjusted for Inflation) from 1986/04 to 2006/03

Table 3-1 gives us the parameter sets for the O-U model for the rate of investment return used in our study. The time unit of these parameter sets is per annum.

Table 3-1 O-U Parameter Sets under Different Choices on Asset Allocation

Asset allocation strategy	α	σ^2	δ	δ_0
A: 100% equity	1.1	0.05	0.06	0.12
				0.06
				0
				-0.06
B: 80% equity 20% long term bond	1.1	0.03	0.057	0.12
				0.06
				0
C: 40% equity 40% long term bond 20% short term t-bill	1.07	0.01	0.04	0.06
				0.04
				0.02
				0
D: 20% equity 40% long term bond 40% short term t-bill	1	0.003	0.03	0.05
				0.03
				0
E: 100% short term t-bill	0.8	0.001	0.02	0.04
				0.02
				0

3.2.2 Parameters for Parametric Mortality Table

Milevsky and Robinson (2000) assumed a Gompertz law for mortality which defines the survival function for a person age x as

$$ {}_t p_x = P(T > t \mid m, l, x) = \exp(\exp(\frac{x-m}{l})(1-\exp(\frac{t}{l}))) \ , \qquad (3.1) $$

where m is the mode, l is the scale parameter and T denotes the random variable for future lifetime of a person age x. They fitted the survival function with the Life Tables, Canada and Provinces 1990-92 (Statistics Canada) and estimated the parameters as

$$m = 81.95, \quad l = 10.6 \quad \text{for males,}$$

$$m = 87.8, \quad l = 9.5 \quad \text{for females.}$$

In our study, we use their estimated mortality tables for comparison purposes and simplicity.

3.2.3 Market Price for Annuitization

According to Milevsky and Robinson (2000), the market price of annuitization for a retiree at age 65 is $14, which could also be considered as a benchmark of wealth to consumption ratio. Insurance firms generally price a life annuity based on the gender[4] and the current interest rate (or some rates related to long-term bond). For females, the price should be higher due to their longevity. If the expected rate of return on long term is higher, the price should be lower due to the higher discount rate from the faster growing asset. For simplicity and comparison, we use the same price of annuitization of $14. We believe it is quite reasonable and will not compromise our goal to address the problems and applications presented in this project.

[4] Some countries prohibited insurance firms from pricing life annuity products based on gender due to non-discrimination issues. In Canada, the regulator has not come to do so. However, many Canadian insurance firms consciously use a blended mortality table to price the life annuity product in order to avoid gender discrimination.

CHAPTER 4: PRESENT VALUE OF A WHOLE LIFE ANNUITY

In this chapter, we will present three approaches to obtain the distribution of the present value of a whole life annuity: the recursive approximate method, the fitted reciprocal gamma method, and simulations. We will use the result from simulations to validate the first two methods.

4.1 Approximate Distribution for the Present Value of a Whole Life Annuity

Parker (1993a) gave a general approach to find an approximate distribution of the present value of future cash flows under stochastic interest rate. In this section, we will use the same approach to find the approximate distribution for the whole life annuity and give the proof in details.

4.1.1 A Recursive Formula

Define the random variable Ξ_n as the (n+1)-year annuity-due certain under the stochastic rate of return, then we have

$$\Xi_n = \ddot{a}_{\overline{n+1|}} = \sum_{i=0}^{n} e^{-Y(i)} = \Xi_{n-1} + e^{-Y(n)} \qquad \text{n=1, …, } \omega\text{-x-1 .} \qquad (4.1)$$

The recursive formula is starting from $\Xi_0 = \ddot{a}_{\overline{1|}} = 1$. Using Parker's method, we define a function

$$g_n(\xi_n, y_n) = f_{Y(n)}(y_n) P(\Xi_n \le \xi_n \mid Y(n) = y_n)$$
$$= P(\Xi_n \le \xi_n) f_{Y(n)}(y_n \mid \Xi_n \le \xi_n). \qquad (4.2)$$

The CDF of Ξ_n can be obtained by

$$F_{\Xi_n}(\xi_n) = \int_{-\infty}^{\infty} g_n(\xi_n, y_n) dy_n . \qquad (4.3)$$

Let Z denote the random variable of whole life annuity, the CDF of Z can be obtained by

$$F_Z(z) = q_x + \sum_{k=1}^{w-x-1} F_{\Xi_k}(z)_{k|}q_x \qquad z \ge 1 . \qquad (4.4)$$

We have an approximate recursive formula for function $g_n(\xi_n, y_n)$ as

$$g_n(\xi_n, y_n) \cong \int_{-\infty}^{\infty} f_{Y(n)}(y_n \mid Y(n-1) = y_{n-1}) g_{n-1}(\xi_n - e^{-y_n}, y_{n-1}) dy_{n-1} . \qquad (4.5)$$

The recursive formula is starting from

$$g_1(\xi_1, y_1) = \begin{cases} \frac{1}{sd(Y(1))} \Phi(\frac{y_1 - E(Y(1))}{sd(Y(1))}) & \text{if } \xi_1 \ge 1 + e^{-y_1} \\ 0 & \text{otherwise} \end{cases} , \qquad (4.6)$$

where Φ (x) is the PDF of the standard normal distribution.

The $Y(n) \mid Y(n-1) = y_{n-1}$ is normally distributed with mean

$$E(Y(n) \mid Y(n-1) = y_{n-1}) = E(Y(n)) + \frac{COV(Y(n), Y(n-1))}{Var(Y(n-1))}(y_{n-1} - E(Y(n-1))) \qquad (4.7)$$

and variance

$$Var(Y(n) \mid Y(n-1) = y_{n-1}) = Var(Y(n)) - \frac{COV^2(Y(n), Y(n-1))}{Var(Y(n-1))} \quad . \tag{4.8}$$

4.1.2 Proof for the Recursive Formula

Again, the recursive formula of $g_n(\xi_n, y_n)$ can be derived with Parker (1993a)'s approach. Starting from

$P(\Xi_n \le \xi_n \mid Y(n) = y_n) = P(\Xi_{n-1} \le \xi_n - \exp(-y(n)) \mid Y(n) = y_n)$, we have

$$g_n(\xi_n, y_n) = P(\Xi_{n-1} \le \xi_n - \exp(-y(n))\, f_{Y(n)}(y_n \mid \Xi_{n-1} \le \xi_n - e^{-y_n}) \tag{4.9}$$

and

$$f_{Y(n)}(y_n \mid \Xi_{n-1} \le \xi_n - e^{-y_n}) = \int_{-\infty}^{\infty} f_{Y(n)}(y_n \mid Y(n-1) = y_{n-1}, \Xi_{n-1} \le \xi_n - e^{-y_n}) \cdot f_{Y(n-1)}(y_{n-1} \mid \Xi_{n-1} \le \xi_n - e^{-y_n})\, dy_{n-1} \,. \tag{4.10}$$

The key point for the approximation is

$$f_{Y(n)}(y_n \mid Y(n-1) = y_{n-1}, \Xi_{n-1} \le \xi_n - e^{-y_n}) \cong f_{Y(n)}(y_n \mid Y(n-1) = y_{n-1}) \,. \tag{4.11}$$

Because of the high correlation between Y(n) and Y(n-1) which is studied in details in Parker (1993a), the approximation is very good. By definition, we have

$$f_{Y(n-1)}(y_{n-1} \mid \Xi_{n-1} \le \xi_n - e^{-y_n}) = \frac{g_{n-1}(\xi_n - e^{-y_n}, y_{n-1})}{P(\Xi_{n-1} \le \xi_n - e^{-y_n})} \quad . \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.10), we obtain

$$f_{Y(n)}(y_n \mid \Xi_{n-1} \le \xi_n - e^{-y_n}) \cong \int_{-\infty}^{\infty} f_{Y(n)}(y_n \mid Y(n-1) = y_{n-1}) \cdot \frac{g_{n-1}(\xi_n - e^{-y_n}, y_{n-1})}{P(\Xi_{n-1} \le \xi_n - e^{-y_n})}\, dy_{n-1} \quad . \tag{4.13}$$

Substituting (4.13) into (4.9), we obtain the final approximate formula (4.5). For the proof of the starting value, we have

$$\Xi_1 = 1 + e^{-Y(1)} \quad . \tag{4.14}$$

Substituting (4.14) into the expression for g_1, we obtain

$$\begin{aligned} g_1(\xi_1, y_1) &= P(\Xi_1 \le \xi_1 \,|\, Y(1) = y_1) \cdot f_{Y(1)}(y_1) \\ &= P(1 + e^{-Y(1)} \le \xi_1 \,|\, Y(1) = y_1) \cdot f_{Y(1)}(y_1)\,. \end{aligned} \tag{4.15}$$

Equation (4.15) can be easily transformed to the expression of (4.6). This completes the proof for the approximate formula for the function of g_n.

4.1.3 Numerical Evaluation

Even if we have the recursive formulas above, it is still hard to evaluate the approximate distribution because of the integration terms in the formulas. Parker (1993a) recommended evaluating those functions numerically with either numerical integration or discretization.

In our study, we use the trapezoidal numerical integration. The function g_n is approximately given by:

$$\begin{aligned} g_n(\xi_n, y_n(i)) \cong \frac{y_{n-1}(nby) - y_{n-1}(1)}{2(nby-1)} \cdot & \\ \{ f_{Y(n)}(y_n(i) \,|\, Y(n-1) = y_{n-1}(1)) \cdot g_{n-1}(\xi_n - e^{-y_n(i)}, y_{n-1}(1)) & \\ + f_{Y(n)}(y_n(i) \,|\, Y(n-1) = y_{n-1}(nby)) \cdot g_{n-1}(\xi_n - e^{-y_n(i)}, y_{n-1}(nby)) & \\ + 2\sum_{j=2}^{nby-1} f_{Y(n)}(y_n(i) \,|\, Y(n-1) = y_{n-1}(j)) \cdot g_{n-1}(\xi_n - e^{-y_n(i)}, y_{n-1}(j)) \}\,, & \end{aligned} \tag{4.16}$$

where nby is the number of equally spaced points between $y_n(1)$ and $y_n(nby)$.

The $y_n(1)$ and $y_n(nby)$ are chosen to be:

$$y_n(1) = E(Y(n)) - 5 \cdot sd(Y(n))$$
$$y_n(nby) = E(Y(n)) + 5 \cdot sd(Y(n)).$$

In our computations, we use nby=100.

Due to the high skewness of the distribution of ξ_n, we have to arbitrarily choose the points of ξ_n. In our program, ξ_n are chosen as 50 equally spaced points between 1 and $E(\Xi_n) + 2sd(\Xi_n)$ and 50 equally spaced points between $E(\Xi_n) + 2sd(\Xi_n)$ and $E(\Xi_n) + 7sd(\Xi_n)$. By doing so, we will have a shorter space between points on the left than that on the right for most parameter sets. The reasons for choosing more points on the left are:

(1) We want more dense points on the left so that we can get a more precise PDF, especially on the left tail.

(2) We want to include points as far as possible on the right tail so that we can have more accurate higher moments later to validate the approximate distribution with those calculated from the exact formula.

The particular values of the function g_{n-1} needed in the above formulas are obtained by linear interpolation:

$$g_{n-1}(\xi_n - e^{-y_n(i)}, y_{n-1}(j)) \cong g_{n-1}(\xi_{n-1}^1, y_{n-1}(j)) + (g_{n-1}(\xi_{n-1}^2, y_{n-1}(j)) - g_{n-1}(\xi_{n-1}^1, y_{n-1}(j))) \cdot \frac{(\xi_n - e^{-y_n(i)}) - \xi_{n-1}^1}{\xi_{n-1}^2 - \xi_{n-1}^1}, \quad (4.17)$$

with ξ_{n-1}^2 being the smallest chosen value for ξ_{n-1} that is larger or equal to $\xi_n - e^{-y_n(i)}$ for which g_{n-1} is known, and ξ_{n-1}^1 being the largest chosen value for ξ_{n-1} that is smaller or equal to $\xi_n - e^{-y_n(i)}$ for which g_{n-1} is known.

Finally, for the $F_{\Xi_n}(\xi_n)$, we have

$$F_{\Xi_n}(\xi_n) \cong \frac{y_n(nby) - y_n(1)}{2(nby-1)}\Big(g_n(\xi_n, y_n(1)) + g_n(\xi_n, y_n(nby)) + 2\cdot\sum_{i=2}^{nby-1} g_n(\xi_n, y_n(i))\Big) \ . \tag{4.18}$$

To calculate the function $F_Z(z)$, we again need particular values of the function $F_{\Xi_n}(\xi_n)$ which can be obtained by linear interpolation.

4.2 Fitting Known Distributions with Exact Moments

Milevsky and Robinson(2000) fitted the distribution of the whole life annuity function under their GBM asset pricing model with some known distributions by using the exact moments. They fitted a reciprocal gamma distribution[5] with the first two moments and Type II Johnson distribution with the first four moments. In our study, we use the first two moments computed under the O-U rate of return model to fit the reciprocal gamma distribution. The parameters for the fitted reciprocal gamma distribution can be calculated as

$$\hat{\alpha} = \frac{2M_2 - M_1^2}{M_2 - M_1^2} \qquad \hat{\beta} = \frac{M_2 - M_1^2}{M_2 M_1} \tag{4.19}$$

[5] Also called inverse gamma distribution.

where M_1, M_2 are the computed moments from the exact formulas (2.17) and (2.18). Here we need to mention that the reciprocal gamma could not accurately describe the real distribution of Z analytically since the real PDF of Z has multiple modes on the left tail with a starting mass probability at 1 while the reciprocal gamma is a smooth single-mode distribution starting at zero. However, it may be still reasonable in the right tail after some moderate values, such as from 10 and up, to study the ruin probability in retirement which may only need to investigate values around 14 – the benchmark value we assumed as the market price of annuitization.

4.3 Simulation

Another method to obtain the distribution of the present value of a whole life annuity is simulation. If the number of simulations is large enough, the empirical distribution from the simulated values should have more credibility than other approximate or fitted distributions. In our study, we will use the percentiles of the distribution from simulations to validate the approximate distribution and the fitted reciprocal gamma distribution.

In our study, we simulate 400,000 lives under different parameter sets of O-U model of the rate of return. For every life, we first generate a realized value for K, the curtate future lifetime, according to the assumed mortality table. Next, we need to generate a path of the rate of return during the retiree's future lifetime. Since the O-U model is a continuous process, we discretize time to small periods to get an accurate approximation of $\int_0^t \delta_r dr$ by $\sum_{i=0}^{nt-1} \frac{1}{n} \delta_{i/n}$ where n=15 is the number of points per year. Then we will get a simulated value z for the present value of a whole life annuity by:

$$z = \sum_{i=0}^{k} pv(i) \ ,$$

where $pv(i) = pv(i-1) * e^{-\sum_{t=(i-1)*n}^{i*n-1} \frac{1}{n}\delta_{t/n}}$ with a starting value of $pv(0) = 1$.

Using the 400,000 simulated values for Z, the whole life annuity, we can estimate an empirical CDF of Z and its moments. We compute the moments with the exact formulas (2.17), (2.18), (2.19) and (2.20) and validate the results from simulations by moments checking. From our results, the first four moments from simulation are very close to the exact values. In most cases, the first four digits of the first two moments are exactly matched and the error of the third and fourth moments are limited to 1% of their exact values. Therefore, we believe that the results from our simulations are accurate enough to describe the real distribution and could be used to validate other methods.

4.4 Validation of the Approximate and Fitted Distributions

Tables 4-1 and 4-2 and Figures 4-1 and 4-2 compare percentiles and moments of the whole life annuity under different parameter sets of the O-U rate of return model for the methods we discussed in previous sections. In the tables, the M1, M2, M3 and M4 mean the first, second, third, and fourth moments.

From these tables and figures, we can see that our recursive approximate method is a very good approximation of the true distribution under the presented parameter sets. In fact, it is true for all parameter sets in Table 3-1 from our program. In Figures 4-1 and 4-2, the approximate distribution is so close to the empirical distribution from simulations that it is hard to tell one from the other. The percentiles are very close even for very high percentiles (99.5% or 99.9%). The first two moments are almost the same as the true values and the third and fourth moments are not too far from the true values. Considering that the moments for the recursive approximate method are calculated numerically, it is hard to eliminate errors from the right tail for higher moments so that we cannot conclude whether the errors on the third and fourth moments are from our approximate method or from the numerical moment calculation algorithm. Interestingly enough, the recursive approximate method for a lower equity allocation strategy is better than a higher equity allocation strategy in terms of higher moments and higher percentiles match. It could be explained from the approximate equation (4.11) which is more accurate for lower equity allocation strategy since Y(n) and Y(n-1) are more highly correlated.

The reciprocal gamma is fitted with the first two moments so that the first two moments will match exactly. However, as we discussed in Section 4.2, the reciprocal gamma cannot provide an accurate fit of the real distribution of a whole life annuity, especially in the tails. The above percentile tables support such claim. There are large errors in the low percentiles and high percentiles of the reciprocal gamma distribution. Interestingly, when we put more assets in equity (see Figure 4-1), the reciprocal gamma becomes better, but is still worse than our recursive approximate method.

Table 4-1 Percentiles and Moments of Approximate Distribution, Fitted Distribution and Simulations for the Present Value of a Whole Life Annuity to a Male Age 65 with O-U Model with α = 1.1, σ^2 =.05, δ = .06, δ_0 =.06

Method	10%	20%	30%	40%	50%	60%	70%	80%	90%	95%	99%	99.5%	99.9%	M1	M2	M3	M4
Approximate Distribution	4.18	5.94	7.21	8.40	9.64	11.05	12.82	15.30	19.77	24.72	38.85	46.33	68.54	11.24	182	4,512	147,438
Reciprocal Gamma	5.31	6.38	7.34	8.32	9.40	10.67	12.30	14.65	18.97	23.85	38.24	46.16	69.98				
Simulations	4.23	6.12	7.42	8.61	9.82	11.19	12.89	15.26	19.46	24.10	37.03	43.53	63.48	11.24	179	4,206	166,487
Exact														11.25	179	4,217	170,574

Table 4-2 Percentiles and Moments of Approximate Distribution, Fitted Distribution and Simulations for the Present Value of a Whole Life Annuity to a Male Age 65 with O-U Model with α = .8, σ^2 =.001, δ = .02, δ_0 =.02

Method	10%	20%	30%	40%	50%	60%	70%	80%	90%	95%	99%	99.5%	99.9%	M1	M2	M3	M4
Approximate Distribution	4.75	7.76	10.13	12.14	13.92	15.60	17.29	19.16	21.63	23.62	27.39	28.77	31.83	13.61	225	4,382	82,231
Reciprocal Gamma	7.65	8.90	9.98	11.04	12.18	13.48	15.08	17.28	21.09	25.10	35.68	40.95	55.31				
Simulations	4.79	7.77	10.16	12.16	13.95	15.62	17.28	19.12	21.50	23.40	26.96	28.34	31.16	13.60	224	4,091	80,366
Exact														13.60	224	4,090	80,378

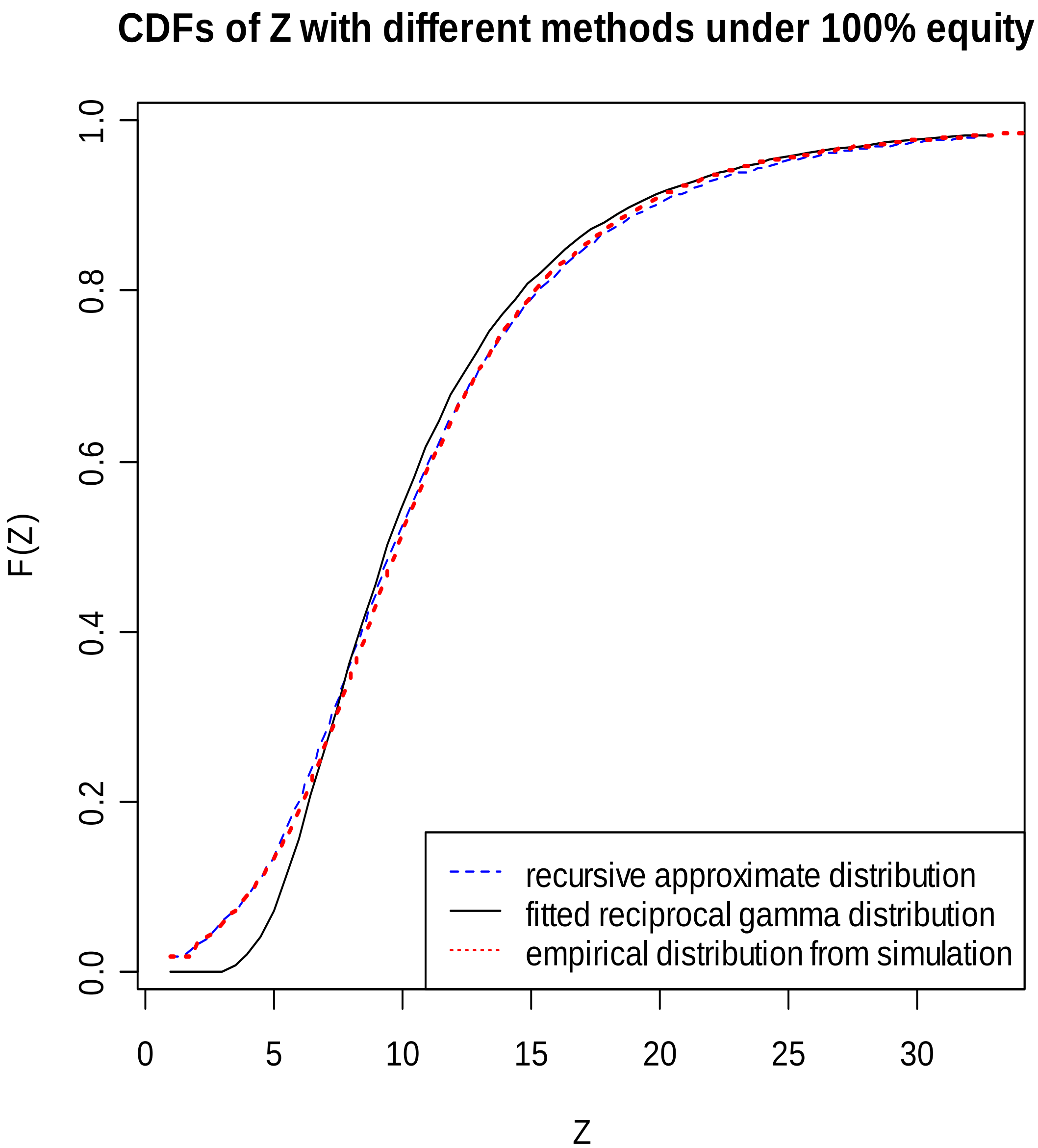


Figure 4-1 The Recursive Approximate Distribution, Fitted Reciprocal Gamma Distribution and Empirical Distribution from Simulations for the Present Value of a Whole Life Annuity to a Male Age 65 with O-U Model with α = 1.1, σ^2 =.05, δ = .06, δ_0 =.06

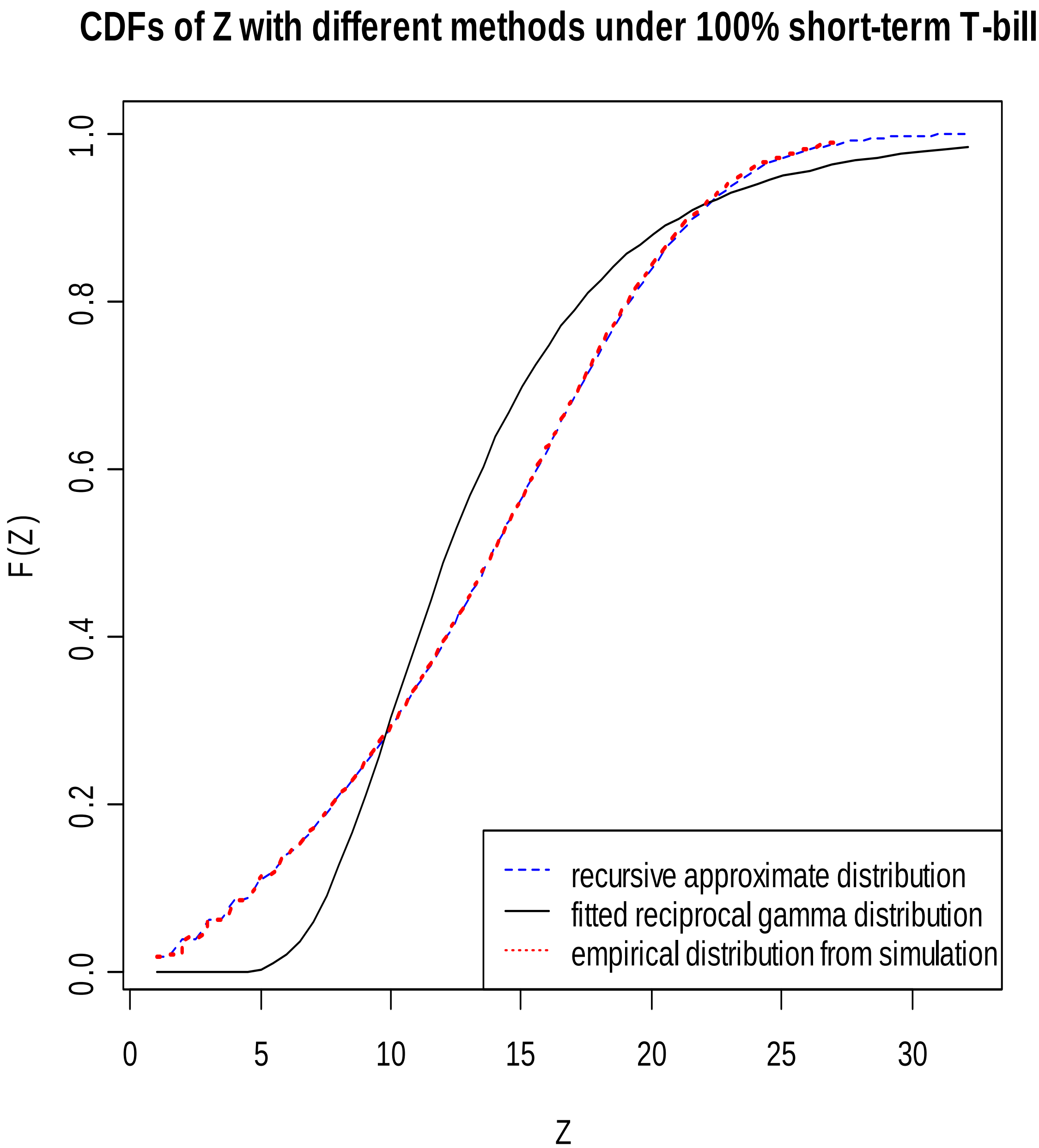


Figure 4-2 The Recursive Approximate Distribution, Fitted Reciprocal Gamma Distribution and Empirical Distribution from Simulations for the Present Value of a Whole Life Annuity to a Male Age 65 with O-U Model with α = .8, σ^2 =.001, δ = .02, δ_0 =.02

CHAPTER 5: RUIN PROBABILITY

In this chapter, we use our approximate distribution of a whole life annuity function to study the ruin probability in retirement under the stochastic rate of investment return and mortality rate.

Here, ruin probability is the probability of running out of money during retirement. Assume a retiree starts with w dollars of wealth and consumes k dollars per period continuously[6]. The net wealth at time t, if the retiree is still alive, under a stochastic return rate, will be

$$\begin{aligned} W(t) &= w \cdot e^{\int_0^t \delta_s ds} - \int_0^t k \cdot e^{\int_s^t \delta_v dv} ds \\ &= e^{\int_0^t \delta_s ds} (w - k \cdot \int_0^t e^{-\int_0^s \delta_v dv} ds) \\ &= e^{Y(t)} (w - k \cdot \int_0^t e^{-Y(s)} ds) \ . \end{aligned} \tag{5.1}$$

We define the ruin probability as $\Pr(\inf_{0 \le t \le T} W_t \le 0 \mid W_0 = w)$. The probability can be expressed in term of the present value of a whole life annuity as follows:

[6] For the derivation, we assume continuous consumption and use continuous whole life annuity function. In our project, we assume consumption or withdrawal at the beginning of each period and use a discrete whole life annuity due. Milevsky and Robinson(2000) gave a similar derivation.

$$
\begin{aligned}
\Pr(\inf_{0\le t\le T}\ W_t \le 0 \mid W_0 = w) &= \Pr(W_T \le 0 \mid W_0 = w) \\
&= \Pr(w - k\cdot\int_0^T e^{-Y(s)}ds \le 0) \\
&= \Pr(\int_0^T e^{-Y(s)}ds \ge \frac{w}{k})\ ,
\end{aligned}
\tag{5.2}
$$

which is the probability that the present value of a whole life annuity exceeds w/k.

From Equation 5.2, assume a male retiree starting with $14 (the market price for a $1 whole life annuity from an insurance firm), self-annuitizes the $14 with $1 consumption per period. Ruin will occur for this retiree if the present value of $1 consumption per period until death is greater than $14. Thus, the ruin probability is equal to the probability that the present value of a $1 whole life annuity, under the asset allocation strategy, will be greater than $14.

5.1 Impact of Asset Allocation Strategy and Initial Rate of Investment Return

Similar to Milevsky and Robinson (2000), we study the impact on the ruin probability of the asset allocation strategy. We also study the impact of initial rate of investment return which is not included in Milevsky and Robinson model. Table 5-1 gives the ruin probability, i.e., Pr (Z>14) where Z is the present value of a $1 whole life annuity under a stochastic rate of return and mortality rate with different parameter sets of O-U rate of investment return model for a male at age 65.

We have the following conclusions from Table 5-1 and Figure 5-1:

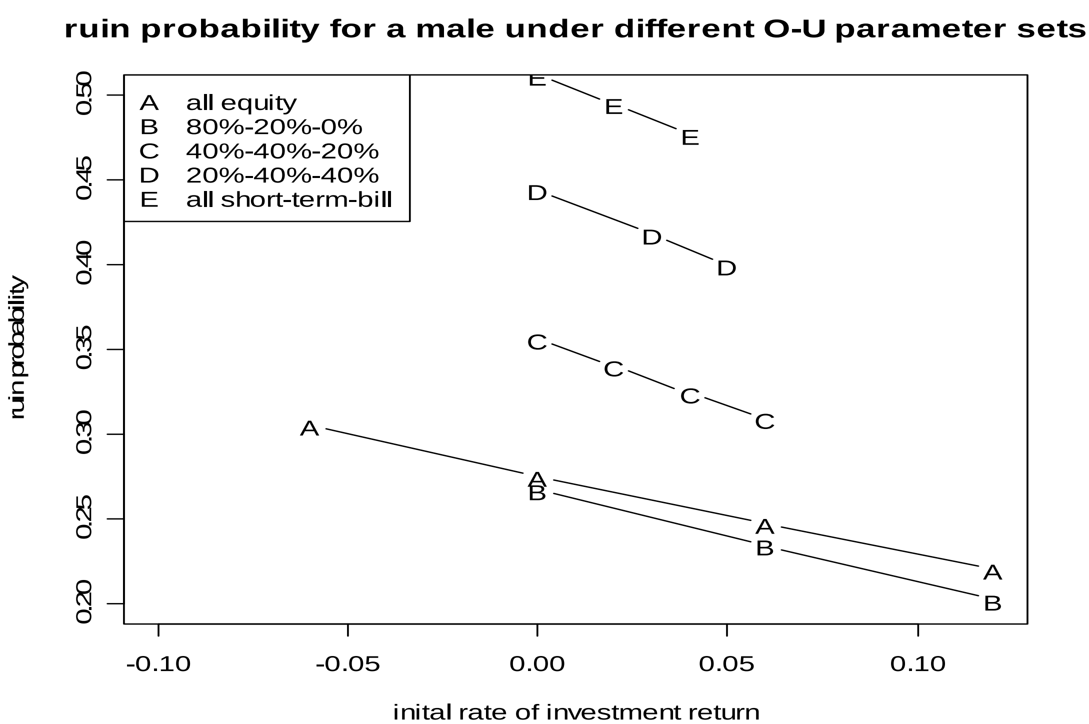
Firstly, the asset allocation strategy is the most important factor for ruin probability. Investing more assets in equity could reduce the ruin probability, but not 100% equity leads to the lowest ruin probability. In our parameter sets, for a male, the asset allocation B (80% equity and 20% long-term bond) has the lowest ruin probability for certain initial rates of return.

Secondly, the initial rate of return has some impacts on the ruin probability, but it is less important than the choice of the asset allocation strategy. If the rate of return at the beginning of retirement is higher, the ruin probability will be lower. It is consistent with a widely accepted wisdom that the first few years' return on investment is crucial to a retirement plan.

Table 5-1 Ruin Probabilities for a Male under Different Asset Allocation Strategies

Asset allocation Strategy	α	σ^2	δ	δ_0	Ruin Probability
A: All equity	1.100	0.050	0.060	0.12	0.220
				0.06	0.247
				0	0.275
				-0.06	0.305
B: 80% equity 20% long term bond	1.100	0.030	0.057	0.12	0.203
				0.06	0.234
				0	0.267
C: 40% equity 40% long term bond 20% short term bill	1.070	0.010	0.040	0.06	0.309
				0.04	0.325
				0.02	0.341
				0	0.357
D: 20% equity 40% long term bond 40% short term bill	1.000	0.003	0.030	0.05	0.401
				0.03	0.418
				0	0.444
E: 100% short term bill	0.800	0.001	0.020	0.04	0.477
				0.02	0.495
				0	0.513

Figure 5-1 Impact of Asset Allocation and Initial Return Rates on the Ruin Probabilities



5.2 Comparison between Males and Females

Assume a male and a female are both starting with $14 for their retirement life and consuming $1 per period, Table 5-2 compares the ruin probabilities between them under different O-U parameter sets.

Table 5-2 Ruin Probability Comparison between a Male and a Female

Asset Allocation	α	σ^2	δ	δ_0	Gender	Ruin Probability
A	1.100	0.050	0.060	0.060	Male	0.247
					Female	0.338
B	1.100	.030	.057	.060	Male	0.234
					Female	0.334
C	1.070	0.010	0.040	0.04	Male	0.325
					Female	0.478
D	1.000	0.003	0.030	0.030	Male	0.418
					Female	0.600
E	0.800	0.001	0.020	0.020	Male	0.495
					Female	0.673

It is obvious that a female will have a higher ruin probability than a male under the same asset allocation option and initial rate of return due to the longevity of females on average.

CHAPTER 6: APPLICATIONS

In this chapter, we use the distribution of the present value of a whole life annuity to study some practical objectives that retirees might have in addition to the consideration of ruin. In Section 6.1, we study the value of bequest in the context of present value and future value. In later sections, we will focus on how much consumption a retiree could have per period (in maximum) according to his own view of risk, the social environment, and his desired standard of living during retirement by using the VaR approach.

6.1 The Value of Bequest

Let Z denote the random variable for the present value of a whole life annuity under our stochastic rate of return model and mortality model, that is, $Z = \ddot{a}_{\overline{K+1|}}$, where K is the curtate future lifetime for a retiree at age 65. Assume a whole life annuity of $1 per year in real term sold by an insurance firm is $14. Further, assume the retiree starts with $14 at age 65. We want to know what the value of the bequest is at the death of the retiree. Clearly, it will be zero if the retiree buys a whole life annuity from the insurance firms with the $14 since the insurance will pay nothing to the retiree's survivors or estate at his or her death. If the retiree self-manages his wealth, he may leave some bequest or nothing in the case when he is ruined before his death. In this section, we will study the present value and future value of his bequest.

6.1.1 Present Value of Bequest

Let W denote the random variable of the present value of the bequest under self-annuitization, then we have

$$W = (14 - Z)_{+} = \max(14 - Z, 0) . \qquad (6.1)$$

Here we assume that if the retiree is ruined in retirement, he will get social assistance or other ways to fully support himself, but will not leave a bequest at death. The expected value of W is

$$E(W) = \Pr(Z = 1) * 13 + \int_{1}^{14} f(z)(14 - z)dz + \int_{14}^{\infty} f(z) \cdot 0dz . \qquad (6.2)$$

The first term is from the probability mass of Z at 1 for those people who die in the first year only receiving (and consuming) the first $1 at the beginning. The second term is the integration on the density function of Z for those people who survive the first year and are not ruined during retirement. The following Tables 6-1, 6-2 and 6-3 illustrate the percentiles of Z from simulations and its mean under different parameters for O-U rate of return model for a male and a female, respectively.

Table 6-1 Present Value of Bequest with Asset Allocation Strategy A (α = 1.1, σ^2 =.05, δ = .06, δ_0 =.06)

CDF	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99	0.999	Mean	Percentage of starting wealth
W for male	0	0	1.11	2.8	4.18	5.4	6.57	7.88	9.75	13	13	4.39	31%
W for female	0	0	0	1.05	2.58	3.91	5.18	6.49	8.26	12.2	13	3.33	24%

Table 6-2 Present Value of Bequest with Asset Allocation Strategy B (α = 1.1, σ^2 =.03, δ = .057, δ_0 =.06)

CDF	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99	0.999	Mean	Percentage of starting wealth
W for male	0	0	1.20	2.63	3.86	5.00	6.17	7.56	9.67	13	13	4.23	30%
W for female	0	0	0	.96	2.27	3.46	4.64	5.96	7.92	13	13	3.11	22%

Table 6-3 Present Value of Bequest with Asset Allocation Strategy E (α = .8, σ^2 =.001, δ = .02, δ_0 =.02)

CDF	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99	0.999	Mean	Percentage of starting wealth
W for male	0	0	0	0	0.06	1.84	3.86	6.25	9.21	13	13	2.76	20%
W for female	0	0	0	0	0	0	0.51	3.06	6.68	12	13	1.62	12%

From the tables above, we have the following conclusions about the bequest in the context of present value:

(1) The present value of bequest has a probability mass at 0 and 13 and continuous density between 0 and 13. The probability mass at 0 is because of the ruin probability and the probability mass at 13 is because of the probability mass of Z at 1 in case of death in the first year.

(2) Comparing results in Tables 6-1, 6-2 and 6-3, by choosing an aggressive allocation strategy (i.e. investing more assets in equity), retirees tend to have a higher present value of bequest left for both males and females. For example, a male retiree will leave 31% of his starting wealth, on average, if all assets are invested in equity market (Table 6-1) while he will leave 20% of his starting wealth, on average, if all assets are invested in short term T-bill (Table 6-3). For a female retiree, choosing all equity could double her wealth on average compared to choosing all short term T-bill.

(3) Males tend to leave more wealth than females. The explanation is that, on average, male retirees die earlier than females which makes males consume less and leave more bequest.

6.1.2 Future Value of Bequest

In practice, it is more natural to think about the value of bequest at the time of death or what we call the future value of the bequest. Let B denote the future value of the bequest in real term at the end of year of death which can be

Table 6-4 Future Value of Bequest (Real Term) with Asset Allocation Strategy A (α = 1.1, σ^2 =.05, δ = .06, δ_0 =.06)

CDF	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99	0.999	Mean	Perc. Of original starting wealth
B for male	0	0	2.37	6.34	9.94	13.2	16.8	24.2	43.6	197	601	20.1	143%
B for female	0	0	0	2.77	7.49	12.1	17.2	27.7	56.3	296	968	25.1	179%

Table 6-5 Future Value of Bequest (Real Term) with Asset Allocation Strategy B (α = 1.1, σ^2 =.03, δ = .057, δ_0 =.06)

CDF	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99	0.999	Mean	Perc. of original starting wealth
B for male	0	0	2.81	6.29	9.36	12.2	14.6	18.8	29.6	103	265	13.75	98%
B for female	0	0	0	2.75	6.77	10.6	14.2	20.0	35.2	142	388	15.00	107%

Table 6-6 Future Value of Bequest (Real Term) with Asset Allocation Strategy E (α = .8, σ^2 =.001, δ = .02, δ_0 =.02)

CDF	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99	0.999	Mean	Perc. of original starting wealth
B for male	0	0	0	0	0.1	2.49	4.92	7.5	10.4	13.3	13.6	3.19	23%
B for female	0	0	0	0	0	0	0.73	4	7.99	13.1	13.5	1.88	13%

expressed as

$$B = (14 - Z)_+ \,/\, pv(k+1) = \max(14 - Z, 0) \,/\, pv(k+1)\,, \qquad (6.3)$$

where pv(k+1) is the present value of $1 at the end of death year.

Similarly, using simulations, we generate Tables 6-4, 6-5 and 6-6 showing the percentiles of B and its mean under different parameters for the O-U rate of return model for a male and a female, respectively.

From these tables, we have the following conclusions about the bequest in the context of future value:

(1) The future value of bequest will have a probability mass at 0 and continuous density above 0. The probability mass at 0 is because of the ruin probability.

(2) Comparing Tables 6-4, 6-5 and 6-6, we observe that the choice of different asset allocation strategies can have a significant impact on the bequest. For example, an all equity asset allocation strategy (in Table 6-4) for a male retiree gives an average bequest of 143% of his starting wealth while an all short-term T-bill asset allocation strategy (in Table 6-6) only leaves approximately 23% of his starting wealth.

(3) Males may leave more or less wealth than females depending on the chosen asset allocation strategy. On the one hand, males live shorter than females which means that males have less time to accumulate the bequest. On the other hand, males have a lower ruin probability than females which increase the chance of leaving a bequest. The former will

dominate the latter under an asset allocation strategy weighted towards equity while the latter will dominate under an asset allocation strategy more heavily weighted towards short term T-bill. Under our all equity asset allocation strategy, females tend to leave a larger bequest than males (Table 6-4) because of their longer lifespan which gives them more time to accumulate more wealth under the higher average return rate on assets. Under our all short-term T-bill asset allocation strategy, females will tend to leave a smaller bequest (Table 6-6) because females will have a greater ruin probability under a lower average return rate on assets and will likely leave no bequest.

6.2 Consumption under Self-Annuitization Using a VaR Method

Suppose a male retiree is risk averse to the possible ruin in the future, but still has some tolerance toward the risk. For example, he could tolerate 5% ruin probability or 50% ruin probability which may depend on some other factors in addition to his own attitude to risk such as:

(1) Social security availability

In most developed countries, there is a social security or benefit system available to give elderly individuals financial assistance if they do not have any income during retirement. For example, Canadian residents are entitled to receive OAS and GIS from the government if they run out of their own money at age 65 or older.

If this kind of social security system is available, many individuals, especially those with a low income may not worry so much about ruin in retirement since the government will have full financial responsibility if they are ruined in retirement. Therefore, people in these countries may have very high tolerance toward ruin, say 50% probability. However, in those countries where the social security or benefit system is not well established, people do not generally expect financial assistance from governments if they run out of money in retirement. Individuals who reside in these countries do worry about ruin in retirement and therefore have a much lower tolerance level, say 5%.

(2) Firm-sponsored or government-sponsored defined benefit pension plan

If there is either a firm-sponsored or a government sponsored defined benefit (DB) plan for a retiree, the wealth under the retiree self-management will generate extra incomes for him in addition to the income from his DB pension plans. The retiree may have a higher tolerance toward ruin than those who do not have, or have a lower pension benefit because he could still enjoy a good retirement life with the income from his DB plans in case of the self-management assets ruin. In Canada, the government sponsored pension plan, CPP, gives retirees about 25% of their final salary at retirement as the retirement benefit. Many firms, particularly government jobs, provide very generous DB plans which give retirees another 35%-45% of their final salary at retirement as the retirement benefit. For individuals with such DB plans, the tolerance toward ruin of other incomes could be very high, say 50% or higher. Note that for retirees with defined contribution (DC) plans, we should regard the wealth in DC plans as the part of assets under our study. The retirees with DC plans have complete responsibility to manage this part of wealth so that they must consider the possibility of ruin on the wealth from DC plans very seriously.

(3) Incentive to leave a bequest to survivors or estate

Some people have a very strong desire to leave wealth to their survivors after death so that they would do their best to avoid ruin in their retirement life. For these people, their tolerance for ruin is very low, say 5% probability or less.

We use the Value-at-risk (VaR) approach to obtain the appropriate price for $1 life annuity that a retiree would like to accept and the maximum consumption per period for a retiree with a certain ruin tolerance. Table 6-7 shows the percentiles of Z, the accepted price and the maximum consumption for a male with different ruin tolerance under an all-equity asset allocation strategy.

From the percentile table of a whole life annuity under all-equity asset allocation strategy, a male with a good DB pension plan who can tolerate 50% probability ruin on his self-managed wealth would accept $9.82 or less for a $1 life annuity while a male without a DB pension plan nor a social security who can only tolerate 5% ruin probability would accept $24.10 or less for a $1 life annuity. If we ignore the expenses and assume that the insurance firms earn returns no higher than individuals, we may say the insurance firms could sell the life annuity as low as $11.24 which is the expected value of $1 life annuity. In our study, we assume insurance firms sell the $1 life annuity at $14 that corresponds to a 24.7% ruin probability. It means that only those retirees with ruin tolerance equal or lower than 24.7% ruin probability will buy the annuity at the price of $14 from insurance firms. In Canada, since most people are covered by social security or firm/government-sponsored DB plans, they may have a much higher ruin tolerance than 24.7% ruin probability which explains why only very few people will annuitize their wealth at retirement.

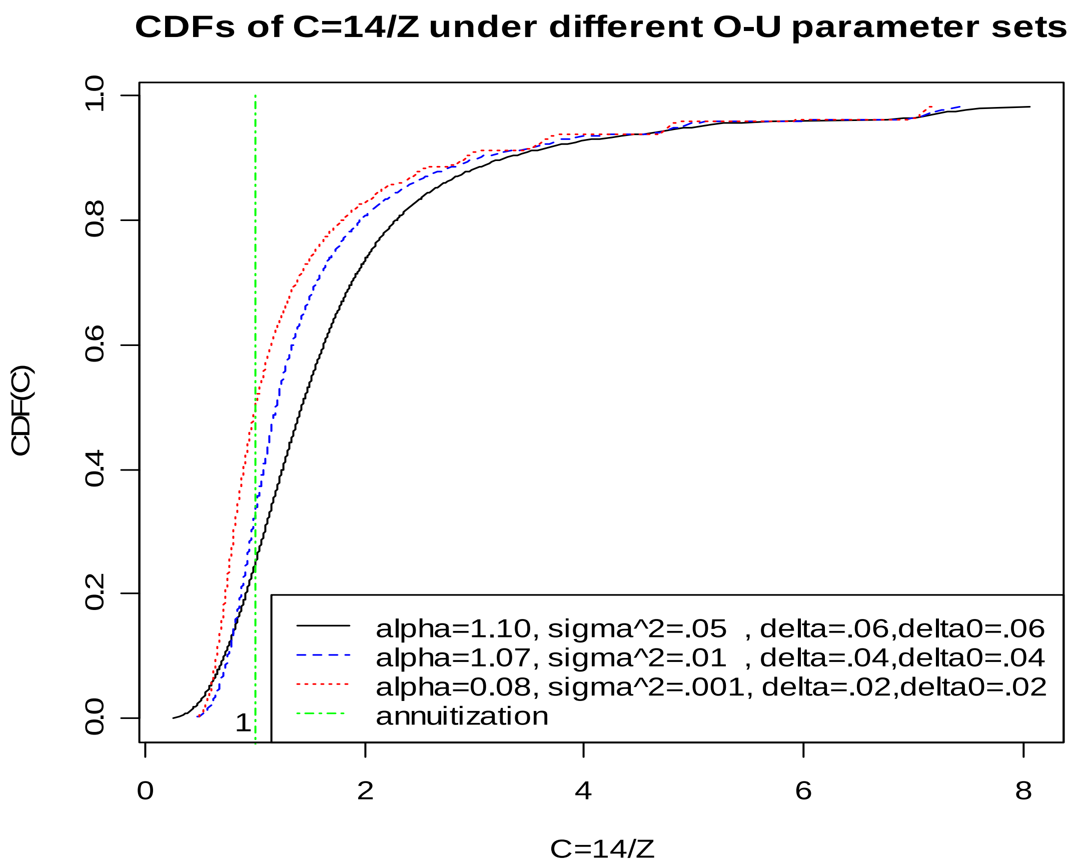
If a male retiree has higher tolerance than 24.7% ruin probability, he should self-annuitize his wealth instead of buying a life annuity from insurance firms priced at $14, so that he will consume more than $1 per period – the amount the insurance firm will provide. For example, if you have a 50% ruin probability tolerance and assume you have X initial wealth, you can consume as much as X/9.82 instead of X/14 per period which is 43% higher.

It could be more straightforward to get the above conclusion by studying the distribution of another random variable defined as C=14/Z. This variable represents the constant consumption per period for which a retiree starting with $14 will just use up all of his wealth upon the time of death. The consumption values are shown in Table 6-7 in the row labelled “Optimal Consumption per Period”. Figure 6-1 shows the cumulative distribution function (CDF) of C under different O-U parameter sets and reveal more interesting points about the optimal consumption.

Table 6-7 Accepted Price for $1 Life Annuity and Maximum Consumption per Period under Certain Ruin Probability Tolerance with Asset Allocation Strategy A (α = 1.1, σ^2 =.05, δ = .06, δ_0 =.06)

F(z)	0.100	0.200	0.300	0.400	0.500	0.600	0.700	.753	0.800	0.900	0.950	0.990
Z	4.23	6.12	7.42	8.61	9.82	11.19	12.89	14	15.26	19.46	24.10	37.03
Ruin Prob. Tolerance	90%	80%	70%	60%	50%	40%	30%	24.7%	20%	10%	5%	1%
Highest price of life annuity accepted by retiree	4.23	6.12	7.42	8.61	9.82	11.19	12.89	14	15.26	19.46	24.10	37.03
Optimal Consumption per period	3.31	2.29	1.89	1.63	1.43	1.25	1.09	1.00	1.00	1.00	1.00	1.00
More consumption than annuitization	231%	129%	89%	63%	43%	25%	9%					
	Self-annuitization							Annuitization				

Figure 6-1 The Distribution of C= 14 / Z under Different O-U Parameter Sets



First, to obtain the optimal consumption as discussed earlier, we use the VaR approach again, but we study the left tail of C instead of the right tail for Z. This can be understood from the fact that Pr(C<c) = Pr(14/Z<c) = Pr(Z> 14/c) = Ruin Probability.

Secondly, C ranges from 0 to 14. C cannot be greater than 14 otherwise the retiree would be ruined at the beginning.

Thirdly, we notice that the lower percentiles of C at the bottom-left corner could be smaller under all-equity asset allocation strategy (black solid line) than the other two asset allocation strategies. It just reflects a widely accepted advice about retirement investment for those with the least risk tolerance that they should reduce or avoid exposure to equity. However, since we assume a retiree could buy a $1 whole life annuity from insurance firms with $14, he will be better off to purchase the whole life annuity instead of considering any asset allocation strategy if his tolerance on ruin probability is below a certain value. From Figure 6-1, it is very clear that the crossover of the CDFs under different O-U parameter sets occurs before $1, which just validates the all-equity asset allocation strategy as the optimal one among the three presented in Figure 6-1 when the tolerance on ruin probability is greater than a certain value for which the corresponding percentile of Z is $1.

Lastly, we notice that the higher percentiles of C at the top-right are very close under different asset allocation strategies. There is a very intuitive explanation that a retiree with much higher tolerance toward ruin in retirement, say over 90% ruin probability, may choose to consume too fast to accumulate wealth in the future. The only no-ruin probability for this situation occurs when the retiree dies too early. Therefore, for retirees with a very high ruin tolerance, the asset allocation strategy may not matter too much.

6.3 Focus on Ruin in the First 10 Years in Retirement

This is a more aggressive consideration for some people, especially lower income people, who may have the desire to consume more only in their first few years after retirement. In Canada, since most people have government social security or firm/government-sponsored DB pension plan, they may have a higher tolerance toward ruin in their later retirement life. Some reasons support this claim:

(1) When retirees are at the beginning of their retirement life, they are still active and very likely to keep their usual living standard but with more free time. They need more money to travel and participate in social events. Therefore, they may have little tolerance toward ruin in their early retirement life.

(2) When retirees become older, they become inactive physically so that they need less money to consume[7]. The income from social security or DB plans may be enough for them to live on. Therefore, they may have very high tolerance toward ruin in their later retirement life.

It may be more appropriate to only focus on ruin in the first few years, say the first 10 years after retirement. Again, we convert this earlier ruin problem to study the distribution of 10-year term life annuity under stochastic interest rate and mortality.

[7] This is especially true for people living in a country like Canada where the government provides a comprehensive health and medical plan to elderly people so that retirees will not be concerned with the rising medical expenses during retirement.

Table 6-8 gives us the percentiles of the random variable

$$Z = \begin{cases} \ddot{a}_{\overline{K+1|}} & K+1 \le 10 \\ \ddot{a}_{\overline{10|}} & K+1 > 10 \end{cases} . \qquad (6.4)$$

From Table 6-8, the 95 percentile of Z gives us the highest accepted price of such 10-year life annuity for a male retiree with 5% probability ruin tolerance in the first 10 year after retirement. If the retiree survives 10 years without ruin, he may choose to annuitize or keep self-annuitize the remaining wealth which may not be a very important issue given that we assume the retiree has decided to rely on the social security or DB plan for the remaining life.

By choosing to focus on ruin in the first 10 years, even a very risk averse retiree, who is starting with $14, with 5% ruin tolerance, can consume $1.16 per period which is 16% higher than the amount provided by a whole life annuity from insurance firms. Only those people who have a ruin probability tolerance in the first 10 years lower than 2% would accept the price of $14 for a whole life annuity. Table 6-8 gives us another strong explanation why Canadians, particularly those with a low income, rarely annuitize their wealth at age of 65.

Note that we discuss two “unfair” values: one is a 10-year term life annuity while the other is a whole life annuity. Nevertheless, if a retiree only focuses on the first 10-year ruin problem, he may not care too much about the difference between a 10-year term life annuity and a whole life annuity. This gives us a strategy to consume more in early retirement and postpone the annuitization decision to a later date.

Table 6-8 Accepted Price for $1 Life Annuity and Maximum Consumption per Period under Certain Ruin Probability Tolerance in the First 10 Year with Asset Allocation Strategy A (α = 1.1, σ^2 =.05, δ = .06, δ_0 =.06)

F(z)	0.100	0.200	0.300	0.400	0.500	0.600	0.700	0.800	0.900	0.950	.981	0.990
Z: 10 year term life annuity	4.18	5.44	6.14	6.70	7.25	7.83	8.51	9.38	10.79	12.11	14.0	15.20
ruin probability tolerance	90%	80%	70%	60%	50%	40%	30%	20%	10%	5%	2%	1%
highest price accepted by retiree	4.18	5.44	6.14	6.70	7.25	7.83	8.51	9.38	10.79	12.11	14.0	15.20
Optimal consumption per period	3.35	2.57	2.28	2.09	1.93	1.79	1.64	1.49	1.30	1.16	1	1
more consumption than whole life annuitization	235%	157%	128%	109%	93%	79%	64%	49%	30%	16%		
	--Self-annuitization---------------------------										annuitization	

6.4 Postponing Annuitization with Minimum Consumption in Later Years

This is an extension of our previous applications. People may argue that retirees may want to consume more in their earlier retirement life, but still need a certain minimum income from their wealth later because the income from social security and government/firm sponsored DB plan may not cover their minimum needs for later retirement. In this application, we design an approach for retirees with such desire. Retirees could postpone annuitization, say 10 year later at age 75, with the remaining wealth in their self-management account while they may consume more in the first 10 years than the amount provided by a whole life annuity from insurance firms at age 65. We require that retirees have enough assets left at age 75 in order to buy a minimum amount of a whole life annuity needed for their remaining retirement life.

First, we have to determine a market price for a $1 whole life annuity at age 75. Reasonably, the whole life annuity at age 75 is less than the whole life annuity at age 65 since one at age 75 will be more likely to receive fewer payments. In our assumption, we only have a market price of $14 for age 65 which is the 75.3 percentile of whole life annuity for age 65 under all equity asset allocation. To make it consistent, we assume insurance firms price the life annuity at the same percentile for age 75. For the all-equity asset allocation, the 75.3 percentile of whole life annuity for age 75 is 10.10, thus we assume it is the market price for a whole life annuity at age 75.

In this application, we redefine ruin in retirement as the situation that either a retiree cannot meet his financial goal to have enough money at age 75 to buy the minimum whole life annuity if he survives over 75, or run out of money if he dies before 75. Define a new random variable as

$$Z = \begin{cases} c_1 \ddot{a}_{\overline{K+1|}} & K+1 \leq 10 \\ c_1 \ddot{a}_{\overline{10|}} + c_2 * 10.10 * PV(10) & K+1 > 10 \end{cases}, \qquad (6.5)$$

where c_1 is the consumption per period in the first 10 years and c_2 is the minimum consumption needed per period which is purchased from insurance firms at the price of \$10.10 at age 75. Assume a retiree starts with \$14 at 65 and desires a minimum \$.50 per period after 75 if he is still alive, we want to know the maximum consumption per period in the first 10 years based on his ruin tolerance (the new definition in this section). We give the percentiles of Z in Tables 6-9 and 6-10 for a male retiree choosing to have c_1=1.19 and c_1 = 1.40 respectively.

Again, using the VaR approach, we can answer the question, "what is the maximum consumption in the first 10 years for a male starting with \$14 under such goal?" If his ruin tolerance is 24.7%, the maximum consumption per period for the first 10 years will be \$1.19 which can be found in Table 6.9. Similarly, if he can tolerate 36% ruin probability, he can consume as much as \$1.40 (obtained from Table 6.10). Notice that we use our new definition of ruin in this section. Table 6-11 shows the value of CDF(14), and the ruin probability= 1- CDF(14) for different consumption levels in the first 10 years by self-annuitization and \$.50 later by annuitization at age of 75.

Table 6-9 Present Value of $1.19 Consumption between Age 65 and 75 and $.50 Annuitization at Age of 75 with Asset Allocation Strategy A (α = 1.1, σ^2 =.05, δ = .06, δ_0 =.06)

F(z)	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.753	0.80	0.90	0.95	0.99
Z	5.00	7.15	8.49	9.59	10.67	11.81	13.15	14	14.92	17.85	20.68	27.54

Table 6-10 Present Value of $1.40 Consumption between Age 65 and 75 and $.50 Annuitization at Age 75 with Asset Allocation Strategy A (α = 1.1, σ^2 =.05, δ = .06, δ_0 =.06)

F(z)	0.10	0.20	0.30	0.40	0.50	0.60	0.64	0.70	0.80	0.90	0.95	0.99
Z	5.86	8.31	9.77	10.98	12.16	13.44	14.00	14.94	20.08	23.22	30.84	33.87

Table 6-11 CDF(14) for the Present Value of Different Consumption Levels between Age 65 and 75 and $.50 Annuitization at Age 75 with Asset Allocation Strategy A (α = 1.1, σ^2 =.05, δ = .06, δ_0 =.06)

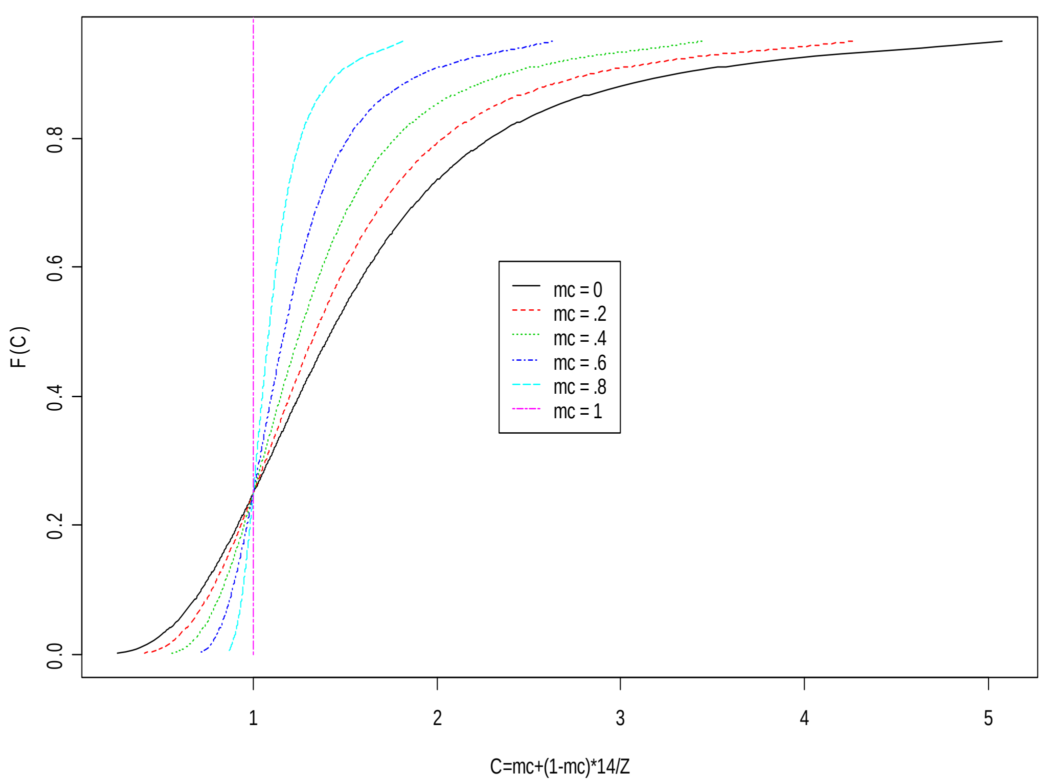
C_1	1.10	1.20	1.30	1.40	1.50	1.60	1.70	1.8	1.9	2.0
CDF(14)	.79	.75	.69	.64	.58	.53	.47	.42	.37	.33
Ruin Prob	.21	.25	.31	.36	.42	.47	.53	.58	.63	.67

6.5 Partial Annuitization

Partial annuitization is a more practical retirement plan chosen by some Canadians. For those people who are very conservative and do not think social security and government/firm-sponsored DB plans will fully meet their financial needs in the future, they may tend to annuitize part of their wealth at the beginning of retirement to guarantee a financially secure retirement. The remaining wealth after annuitization will give them supplemented income if they manage it well. Here, we redefine ruin in retirement as the situation that retirees run out of their own self-management wealth after partial annuitization. Assume a male retiree starting with $14 at 65 wants to have a guaranteed minimum income of $.50 to supplement his social security and DB pension plan. At the beginning, the retiree has to take out $7 to purchase a $.50 life annuity from insurance firms and the remaining $7 will provide the additional income to his minimum consumption – social security + DB plan + $.50. Using the same approach presented in Section 6.2, we can calculate the maximum consumption per period provided by the remaining $7. For example, from Table 6-1, a male retiree with 50% ruin probability tolerance who invests all self-managed asset in equity would like to consume as much as $1.43*(7/14)=$.715 per period from the remaining $7. In other words, the retiree starting with $14 at 65 could have a guaranteed consumption per period – social security + DB plan + $.50 plus a non-guaranteed $.715 per period which will have a probability of 50% not lasting during his entire retirement life.

Similarly, we can reach the above conclusion by looking at the distribution of another random variable defined as C= mc + (1- mc) * 14 / Z where mc is a constant between 0 and 1. This random variable represents the constant consumption per period under a retirement plan consisting of annuitizing mc*100% of the wealth at the beginning of retirement and self-managing the remaining which will be just used up at the time of death. We draw the CDFs of C for mc=0, .2, .4, .6, .8, 1 under the all-equity parameter set for O-U model in one graph which will reveal some interesting points.

Figure 6-2 CDFs of C= mc + (1-mc)*14/Z for mc=0, 0.2, 0.4, 0.6, 0.8, 1.0 under Asset Allocation Strategy A (α = 1.1, σ^2 =.05, δ = .06, δ_0 =.06)



First, C ranges from mc to mc+ (1-mc)*14. The lower range of C will be equal to mc, the guaranteed part provided by partial annuitization. From Figure 6.2, we can see that the left tails of the CDFs of C start at mc instead of 0. C cannot be greater than mc + (1-mc) * 14 because the retiree will be borrowing at the beginning or be ruined under the new definition of ruin in this section.

Secondly, mc=0 corresponds to the100% self-annuitization case while mc=1 is the 100% annuitization case; the others are between these two extreme cases. There is only one crossover point among these CDFs which occurs at C=1. From Figure 6.2, at the bottom left corner, the case mc=1(i.e., 100% annuitization), will have the highest value for the same percentile which is always equal to $1 while at the top right corner, the case mc=0 (i.e., 100% self-annuitization), will have the highest value for the same percentile. If we only use the VaR approach to make a choice, we could have the decision of either 100% self-annuitization or 100% annuitization and any partial annuitization will be suboptimal. It is obvious that the VaR approach could not correctly reflect the process of decision-making for retirees who choose partial annuitization. Perhaps utility functions may be more appropriate to explain the decision to partially annuitize.

CHAPTER 7: CONCLUSION

Milevsky and Robinson(2000) studied the ruin problem in retirement under a standard Geometric Brownian Motion asset pricing model, or equivalently a stochastic model on force of interest accumulation function as $Y(t) = \delta \cdot t + \sigma \cdot B_t$ where B_t is a geometric Brownian motion process. Compared to their interest model, we model the rate of return as an O-U process which will give a more detailed description of the interest process.

We obtained an approximate distribution for a whole life annuity by a recursive approach introduced in Parker(1993a) which is better than the reciprocal gamma distribution estimated by the moment matching method in terms of percentile matching, particularly for low percentiles and high percentiles, under various asset allocation strategies.

For the ruin probability, we get very similar results to those found in previous works. First, investing in more equity will reduce the ruin probability, but the optimal asset allocation strategy is not 100% equity. Secondly, females tend to have a higher ruin probability under the same situation. In addition, because our O-U model includes the initial return rate, we can study its impact on the ruin problem in retirement and we conclude that the return in the first few years is crucial.

Under our O-U rate of return model, we study the value of the bequest. We get similar results to Milevsky et al. (1997). More equity exposure will help a retiree leave a higher value of bequest both in the context of the present value and the future value. In the context of future value, females will leave higher value of bequest than males with more assets in equity while lower value with less assets in equity. Then we obtained the optimal/maximum consumption per period for retirees with different tolerance to ruin probability, social environment, and standard of living in retirement by the VaR approach. The general idea is to help retirees achieve a maximal, but sustainable, consumption if they can match their objectives and tolerance levels with one of our cases.

Under our O-U rate of return model, we will be able to explore more things in the future. First, it is very important to find a general and simple approach to estimate the O-U model, not only based on the past market data but also incorporating the economic forecast. Secondly, for the market price of annuitization, a single fixed rate is obviously not practical. We could add another stochastic process to model the interest rate at which insurance firms price the life annuity so that we can determine a more appropriate market price according to the current rate or the forecast on the long-term bond rate. Lastly, in order to give more practical and realistic advice on retirement problems, we should add taxation into the model.

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