IBNR Claims Reserving using a Compound Poisson INAR(1) Model

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Abstract

In non-life insurance, reserving models for incurred but not reported (IBNR) claims are extensively studied for estimating and predicting adequate reserves for the company. We present a Poisson integer-valued autoregressive (INAR) model of order one for closed claim counts and a compound model based on it in which a mixed gamma distribution for claim severities is assumed. The compound model we study takes into account both the IBNyR (incurred but not yet reported) and IBNeR (incurred but not enough reported) claim counts and their payments. Maximum likelihood techniques are applied for estimating the model parameters. The simulation study is adopted to illustrate the results of the estimations, and to compare the performance of different sizes of the loss development triangle. Predictions based on our proposed model are discussed and the level of estimation accuracy is examined.

Keywords: IBNR; INAR; Maximum likelihood estimation; Loss triangle; Compound model; Aggregate claims

Dedication

To my beloved parents, for their unconditional and endless support.

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Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

In the insurance context, the claim reserve and loss reserve are equivalent, and a reserve for claims or losses is basically the total amount of money that needs to be put aside for payments associated with future claims or losses. Loss reserves represent one of the major liabilities of an insurance company, and hence are important components for companies' financial evaluation. Studies on modeling claims and predicting reserves have been conducted extensively for past years. The loss reserves in non-life insurance can be divided into two big components which are Incurred But Not yet Reported (IBNyR) and Incurred But Not enough Reported (IBNeR) (see, for example, Friedland, 2010 for more details). IBNyR claims arise from the claims which are yet to be reported whereas IBNeR claims are incurred from the claims that have been already reported. In general, Incurred But Not Reported (IBNR) claims refer to IBNyR claims and IBNeR claims. IBNyR and IBNeR claims are also known as pure IBNR and RBNS (Reported But Not yet Settled) claims, respectively in the literature. Estimating loss reserves can be done without distinguishing their types; however, loss reserves can also be estimated as a sum of two separate estimates of IBNyR and IBNeR claims, which may improve the level of accuracy in estimations.

Organizing IBNR claims data is the first and an important step for estimating loss reserves. In practice, the loss development triangle (also known as run-off triangle) is a commonly used method to organize IBNR claims data. As these claims are not paid immediately, there are development periods for claim payments. For example, in automobile insurance, typical loss development triangle organizes the claims data by the year that the accident occurs, called the accident year, and the year that the claims are paid or settled, called the development year. In this way, changes or developments in losses between different times of evaluation can be estimated.

Various methods and models, both non-parametric and parametric, have been studied and proposed to estimate loss reserves. One of the most popular and classical methods for estimating the IBNR claims is the Chain-Ladder (CL) method because of its simplicity. However, the CL method has a disadvantage of no indication of the variability of results (see, Mack, 1993). To improve such disadvantage of the classical model, the Double Chain-

Ladder (DCL) method is proposed in Martinez-Miranda et al. (2012). The DCL method is formulated in a way to acknowledge the data being used and is capable of dividing predicted claims reserve into that of IBNR and RBNS using a simple regression approach. Another disadvantage arisen from the CL method is that the tail estimates are not accurate. However, by incorporating the method proposed in Bornhuetter-Freguson (1972) with the DCL method, the tail estimates can be improved (see, Martinez-Miranda et al., 2013). These popular methods have a strength of simplicity and have been extensively used in property and casualty insurance practice. However, as they are non-parametric models and not studied under the stochastic framework, the uncertainty of the estimators is not assessed and tested. Thus, parametric models have been studied and developed for the claim reserving. Claim counts and their sizes can be studied with separate models, and then the total reserve can be estimated by multiplying the estimated ultimate number of claims and the estimated ultimate size of claims obtained from these two models; this is referred to as the frequency-severity techniques. The advantage of this technique is that it can provide potential insight into the claims process (Friedland, 2010). Thus, it can be used as a reasonable start point to study claim reserve in practice. Another way to study parametric models for claim reserving is to model the aggregate claims with payments using a compound model such as Tweedie's model (see, Wüthrich 2003).

One of the well studied claim reserving models is the Tweedie's model presented in Jørgensen and de Souza (1994) where a Tweedie's compound Poisson model is fitted to insurance claim data for tarification. Wüthrich (2003) further presents a model for the normalized incremental payments from the aggregated data by reparameterizing Tweedie's model. In fact, Tweedie's model is a special compound model in which the number of payments follows a Poisson distribution, and sizes of payment follow a Gamma distribution. Moreover, Tweedie's model has been demonstrated to belong to the exponential dispersion family with variance function $V(\mu) = \mu^p, p \in (1, 2)$ and dispersion parameter ψ by a re-parameterization; see, for example, Jørgensen (1987) for more details about exponential dispersion model. With a constant dispersion parameter, the class of Tweedie's model consists of three models, over-dispersed Poisson ($p = 1$), gamma model ($p = 2$) and compound Poisson. Details can be found in Wüthrich (2003).

To improve the estimations from the aggregation of data, a stochastic model for loss reserving is proposed by Verrall et al. (2010) using the run-off triangle of paid claims and also the number of reported claims. In the study, separate models for IBNeR and IBNyR claims are derived in terms of the different sources of delays which allow separate prediction of IBNeR and IBNyR claims. In their paper, it denotes the number of claims paid as $N^{paid}_{i,j,k}$, as the part of $N_{i,j}$ number of claims incurred in period i with j periods delay to be reported and with k periods of delay to be fully paid. When $k = 0$, this is the number of claims being fully paid in the same period as they are reported. The aggregate number of paid claims is denoted as

$$N_{i,j}^{paid} = \sum_{k=0}^{\min(j,d)} N_{i,j-k,k}^{paid},$$

where d is the maximum periods for IBNeR delay which can be determined from the data. The two different types of delay, IBNyR delay and IBNeR dealy, are modeled separately by assuming that $N_{i,j}$ are independently distributed with an over-dispersed Poisson distribution, and $N_{i,j,k}^{paid}|N_{i,j}$ are multinomially distributed. Based on this model for number of paid claims, the paper further presents a model for aggregate incremental claims as

$$X_{i,j} = \sum_{l=1}^{N_{i,j}^{paid}} Y_{i,j}^{(l)},$$

where $Y_{i,j}^{(l)}$ denotes the l^{th} individual claim payment which can be zero.

Although models for claim reserving have become more sophisticated and developed due to extensive studies, the time series approaches have not been widely employed to model the claim counts. Kremer (1995) proposes an integer-valued autoregressive (INAR) of order one model for IBNR claims based on the INAR(1) process proposed in Al-Osh and Alzaid (1987). More insights and properties of this model as well as methods for estimating model parameters can be found in Al-Osh and Alzaid (1987).

The use of INAR(1) process for modeling claim counts in risk models presented in Cossette et al. (2010). The paper considers an insurance portfolio, and defines the aggregate claim amount in period k as

$$W_k = \sum_{j=1}^{N_k} B_{k,j},$$

where N_k represents the number of claims in period k, and $B_{k,j}$ represents the j^{th} claim size in period k. To incorporate the temporal dependency, they propose an INAR(1) process for the counts described by

$$N_k = \alpha \circ N_{k-1} + \epsilon_k,$$

where $\underline{N} = \{N_k, k \in \mathbb{N}^+\}$ is a Poisson autoregressive of order one process with N_1 following a Poisson distribution with mean λ, and $\underline{\epsilon} = \{\epsilon_k, k \in \mathbb{N}^+\}$ is a sequence of independent and identically distributed (i.i.d.) r.v.'s following a Poisson distribution with mean $(1-\alpha)\lambda$ and $\alpha \in [0, 1]$ (detailed definition of INAR(1) process is given in Chapter 2). This model can be interpreted as a sum of the population who arrives in the interval $(k-1, k)$, and the population who survives from $k-1$ to k. Further, the paper also presents the properties such as probability and moment generating functions, and an expression for the Lundberg coefficient under the assumption that claim sizes are exponentially distributed to illustrate the impact of the dependence parameter α to the risk model. In Cossette et al. (2011), the same model is proposed to study the stop-loss premium, value at risk (VaR), and tail value at risk (TVaR).

Based on a Kremer's INAR(1) model for IBNR claims, a Poisson integer-valued autoregressive model for number of the unclosed claims is proposed by Bai (2016). In the study, the model parameters are estimated using three non-parametric methods (1) Yuller-Walker estimation, (2) Conditional Least Squares estimation (CLS) and (3) Iterative Weighted Conditional Least Square estimation (IWCLS). The mean square error prediction is used for the prediction inference. The assumptions and properties of this model are briefly reviewed in Chapter 2.

As the number of claims have the temporal dependence, we propose a model for number of the closed claims using INAR process of order one developed from the idea proposed in Bai (2016). We further study a compound model with INAR(1) for counts and a mixed gamma distribution for individual claim sizes. Our model for closed claim counts takes into account both the IBNyR and IBNeR claims, which can be used to predict the number of claims expected to be paid out in the future. By incorporating the number of closed claims with sizes of claim, our compound model can be utilized to estimate the total amount of future payments which is a major interest of the company's perspective. We apply maximum likelihood techniques for estimating the model parameters. As the proposed compound model is a parametric model and studied under the stochastic framework, the uncertainty in model parameter estimations can be assessed. A closed rate is introduced in the model for closed claims count. When the closed rate is equal to 1, all the claims are closed with payments when they are being reported. In this case, our INAR(1) model for closed claims counts reduces to a Poisson model, and our proposed compound model is found to be Tweedie's compound Poisson model. Thus, our proposed compound model is considered as a generalization of Tweedie's model. In many cases, data on number of and sizes of claims are available; thus, the proposed compound model can make more efficient use of claims data. We conduct a simulation study to illustrate the parameter estimations and their accuracy level, and to analyze the prediction results. The mean square prediction error has been used as a measure for evaluating prediction errors.

The outline of the project is as follows. In Chapter 2, we introduce the basics of loss development triangle and propose models for closed claim counts and incremental individual claim sizes. Properties of these models are discussed. Chapter 3 presents the procedure to apply the maximum likelihood estimation methods. The systems of estimating equations are derived. In Chapter 4, an algorithm for estimating parameters is presented, and the prediction method is introduced. Chapter 5 provides the numerical illustration from the simulation study and the analysis on the predictions. Chapter 6 concludes the project with a discussion of further research.

Chapter 2

Model for IBNeR and IBNyR Claims

The IBNR claims reserving model is mainly used to study the loss development trend or pattern using the readily available data so the payments for future claims can be well reserved to reduce the financial loss of the insurance company. The IBNR data is normally organized by the loss development triangle that is introduced briefly in Section 2.1. As we have mentioned in Chapter 1, the INAR(1) model for closed IBNR claim counts proposed in this project is developed from a similar model studied in Bai (2016) for unclosed claim counts. Before we introduce our model, in Section 2.2, we present a brief review of the model and its assumptions for the unclosed IBNR claim counts. We then present in Section 2.3 our count model for closed IBNeR and IBNyR claims, model assumptions and properties. In Section 2.4, models for loss severities are introduced and their properties are presented. Finally, Section 2.5 presents a compound model for incremental aggregate claims (total payments) based on the counts and severities models discussed in previous two sections. Some characteristics of the compound model are also discussed.

2.1 Loss development triangle

The loss development triangle is a commonly used actuarial technique to arrange the IBNR loss data from the past experience. The most recent accident year and the latest development year are denoted as I and J, respectively. In general, I and J can be different depending on the type of the loss data. However, we use $I = J$ in this project for simplicity. The general loss development triangle, which separate claim figures on two time axes, is illustrated in Figure 2.1. In this figure, $W_{i,j}$ can be claim numbers or payments and these figures can be either incremental or cumulative in accident year i with reporting delay of j years. The observed data is recorded into upper left triangles of the table by the accident year and the development year while lower right triangle of the table contains the predictions/estimations of corresponding quantities. For the individual claim sizes which are studied in Section 2.4,

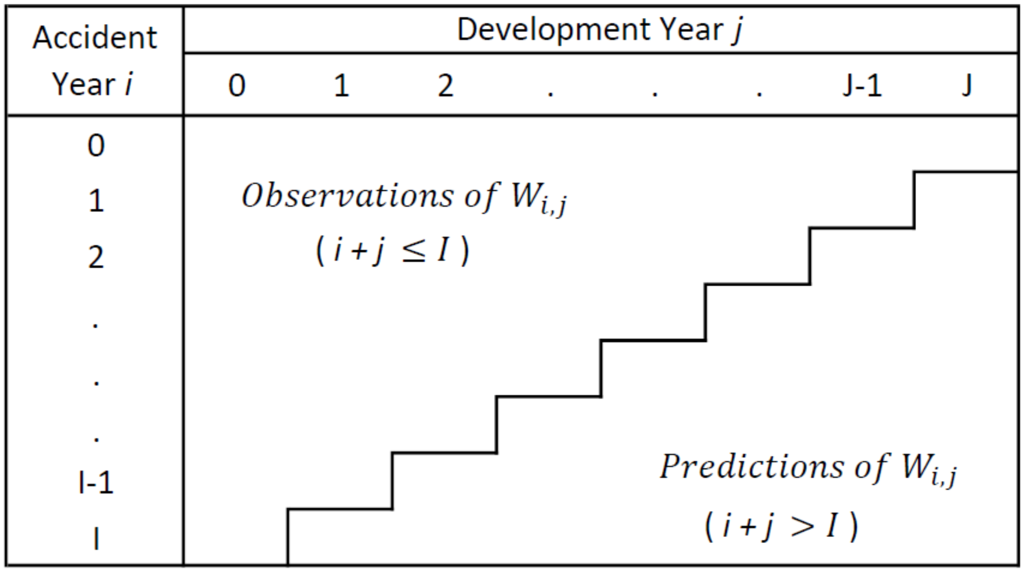

Figure 2.1: Loss development triangle

each cell (i, j) in the upper left triangle contains the number of observed individual claim sizes and in the lower right triangle contains predictions/estimates of individual claim sizes.

2.2 Model for Unclosed Claims

We now review the model proposed in Bai (2016) for unclosed IBNR claim counts. For the unclosed claim counts, the idea of constant unclosed rate is introduced, and a Poisson INAR(1) model is assumed. Define $N_{i,j}$ as the total number of claims that occur in accident year i and have been reported up to development year j but have not yet been settled at the end of the development year j. Assumptions for $N_{i,j}$ model are as follows:

- Unclosed claims $N_{i,j}$ of different accident years i are independent (i.e., $N_{i,j} \perp N_{l,k}$ for any j and k and $i \neq l$).
- There exist parameters $\mu_0, ..., \mu_I$ and $\gamma_0, ..., \gamma_I$ such that newly reported claims $I_{i,j}$ incurred in accident year i but reported with j years of delay are independently Poisson distributed with $\mathrm{E}[I_{i,j}] = \mu_i \gamma_j$, for all $0 \leq i, j \leq I$, and $\sum_{j=0}^{I} \gamma_j = 1$.

In Bai (2016), it is proposed that the unclosed number of claims $N_{i,j}$ of accident years i and development year j follows an INAR(1) process such that

$$N_{i,j} = \rho \circ N_{i,j-1} + I_{i,j}, \quad 0 \leq i, j \leq I, \tag{2.1}$$

where $\rho \circ N_{i,j-1} = \sum_{k=1}^{N_{i,j-1}} Y_k$, $Y_k \overset{iid}{\sim} Bernoulli(\rho)$ and $0 \leq \rho \leq 1$, and $N_{i,-1} = 0$. Here, "$\circ$" is called the binomial thinning operator. We can interpret (2.1) as the sum of unclosed claims carried from previous year $j-1$, and newly reported claims in year j.

2.3 Closed Claims Counts

The number of closed claims can be seen as the difference of a total number of outstanding claims in two consecutive development years (say, $j-1$ and j). By applying this idea to unclosed claims, the model for closed claims is derived, and some assumptions presented for unclosed claims are applied to the model. It turns out that the model for the number of closed claims is also a Poisson INAR process of order one. Section 2.3.1 lists assumptions for the closed claims model. Section 2.3.2 presents the characteristics such as conditional moments, unconditional moments and distribution of closed claims.

2.3.1 Model Assumptions

In the Poisson INAR model proposed by Bai (2016), ρ is interpreted as the unclosed rate. In other words, we can interpret $1-\rho$ as the closed rates which can be used to model the closed claims with the INAR process. We now define the closed claims $R_{i,j}$ as the total number of claims occurred in accident year i that incurs payments in development year j.

Assumption 2.3.1.

- The newly reported claims in accident year i and development year j are not closed within the same development year
- Closed claims $R_{i,j}$ for different accident year i are mutually independent from each other
- Each claim is settled with a single payment; there are no such claims being paid in partial payments.

Under Assumption 2.3.1, we have zero number of claims when $j=0$, and the closed claims $R_{i,j}$ can be defined as

$$R_{i,j} = N_{i,j-1} - \rho \circ N_{i,j-1} = (1-\rho) \circ N_{i,j-1}, \quad 0 \leq i, j \leq I \tag{2.2}$$

In (2.2), $R_{i,j}$ can be interpreted as the total number of claims from the last development year carried to the current year at closed rate $(1-\rho)$. Note that by using (2.1) and (2.2), $R_{i,j}$ has the following decomposition:

$$\begin{aligned}
R_{i,j} &= (1-\rho) \circ N_{i,j-1} \\
&= (1-\rho) \circ (\rho \circ N_{i,j-2} + I_{i,j-1}) \\
&= (1-\rho) \circ \rho \circ N_{i,j-2} + (1-\rho) \circ I_{i,j-1} \\
&= \rho \circ R_{i,j-1} + (1-\rho) \circ I_{i,j-1}, \quad 0 \leq i, j \leq I,
\end{aligned} \tag{2.3}$$

where $I_{i,j-1}$ is the newly reported claims $I_{i,j-1}$ for accident year i and development year $j-1$. That is, the total number of closed claims $R_{i,j}$ is the sum of number of reopened claims in year $j-1$ that have (incurred) payments in development year j, i.e., IBNeR claims, and the number of newly reported number of claims in year $j-1$ that are closed in year j at the rate of $1-\rho$ i.e., IBNyR claims. Moreover, from (2.3) we can see that $R_{i,j}$ also follows an INAR(1) process with innovations follow a compound Poisson distribution with the following additional assumptions:

Assumption 2.3.2.

- $I_{i,j-1}$ for $0 \leq i \leq I$, $1 \leq j \leq I+1$ are independently Poisson distributed with $\mathrm{E}[I_{i,j-1}] = \mu_i \gamma_{j-1}$, and there exist parameters $\mu_0, \mu_1, ..., \mu_I$ and $\gamma_0, \gamma_1, ...\gamma_I$ such that $\sum_{j=0}^{I} \gamma_j = 1$

- The closed claims $R_{i,j}$ follows an INAR(1) process as proposed in (2.3) with $\rho \circ R_{i,j-1} = \sum_{k=1}^{R_{i,j-1}} Y_k$ and $(1-\rho) \circ I_{i,j-1} = \sum_{k=1}^{I_{i,j-1}} Z_k$ follows Compound Poisson distribution, where Y_k and Z_k are Bernoulli distributed with mean ρ and $(1-\rho)$ respectively.

- Z_k and $R_{i,j-1}$ are independent to $I_{i,j-1}$

- $R_{i,-1} = 0$ and $I_{i,-1} = 0;$ thus, $R_{i,0} = 0$ and $R_{i,1} = (1-\rho) \circ I_{i,0}$

Note that the definition of thinning operator $\rho \circ (1-\rho) \circ R = \rho \cdot (1-\rho) \circ R$ (see, Ristić, 2013). We now give following remarks regarding the model we have proposed.

Remark 2.3.1.

- Although the claim settlements may often involve more than one payments or annuity-type payments in practice to settle the claims, we assumed that the claims are closed/settled with one payment per claim for the sake of simplicity. Also, we believe that the studies under this assumption should provide some useful insights for further investigations.

- The mean of newly reported claims in accident year i and development year $j-1$ is the product of μ_i and γ_{j-1} in Poisson INAR(1) process for closed claims. Similar to the interpretation presented in Bai (2016), μ_i is the total expected number of claims in accident year i and γ_{j-1} is the fraction of the claims reported in development year $j-1$.

- In this project, a closed rate ρ is assumed to be constant for the purpose of simplicity. However, this may not be the reasonable assumption in practice as there are many other factors varying the number of closed claims. The large portion of total number of claims are usually settled quickly, and the complicated cases may take longer to get settled which implies that the closed rate ρ may depend on development year

j. For example, the complicated case often involve litigation which requires a longer period to be settled. Also, there are 1) the change of line of business, 2) the change of environmental factors, 3) the change of technology which may require different closed rates for different accident years.

- We assume that the newly reported claims are not closed within the same year; although, this may not be always true. The short tail business such as property insurance is an example of which losses are settled within a relatively short period of time. For this case, the estimation of claim counts may be underestimated by not including the newly reported claims which are settled within the same year.

2.3.2 Properties

The proposed Poisson INAR(1) model for closed claims has following properties.

Proposition 2.3.1. The closed number of claims $R_{i,j}$ can be written as a summation of newly reported claims in the past development years $j-k$, $1 \le k \le j$, that is,

$$R_{i,j} = \sum_{k=1}^{j} \left((1-\rho)\rho^{k-1}\right) \circ I_{i,j-k}, \quad 0 \le i, j \le I \tag{2.4}$$

where $R_{i,0} = 0$

Proof. From (2.2), we have

$$\begin{aligned} R_{i,j} &= (1-\rho) \circ N_{i,j-1} \\ &= (1-\rho) \circ (\rho \circ N_{i,j-2} + I_{i,j-1}) \\ &= (1-\rho) \circ (\rho \circ (\rho \circ N_{i,j-3} + I_{i,j-2}) + I_{i,j-1}) \\ &= (1-\rho) \circ (\rho^2 \circ N_{i,j-3} + \rho \circ I_{i,j-2}) + I_{i,j-1}) \\ &\vdots \\ &= \sum_{k=0}^{j-1} (1-\rho) \circ \rho^{k-1} \circ I_{i,j-(k+1)} \\ &= \sum_{k=1}^{j} (1-\rho) \circ \rho^{k-1} \circ I_{i,j-k}. \end{aligned}$$

Noting the fact that

$$(1-\rho) \circ \rho^{k-1} \circ I_{i,j-k} \stackrel{d}{=} \left((1-\rho)\rho^{k-1}\right) \circ I_{i,j-k},$$

where notation "$\stackrel{d}{=}$" means that both sides follow a same distribution, we then obtain (2.4). □

According to (2.4), the closed claims $R_{i,j}$ can be expressed as the summation of closed claims from all the newly reported claims with the delay of $1, 2, ..., j-1$ years. The probability of the newly reported number of claims with delay $j-k$ being closed in year k is $(1-\rho)\rho^{k-1}$. Moreover, the closed claims $R_{i,j}$, given the total number of unclosed claims $N_{i,j-h}$, can be rewritten as

$$R_{i,j} = \sum_{k=1}^{h-1} \Big((1-\rho)\rho^{k-1}\Big) \circ I_{i,j-k} + \Big((1-\rho)\rho^{h-1}\Big) \circ N_{i,j-h}, \tag{2.5}$$

and the closed claims $R_{i,j}$, given the total number of closed claims $R_{i,j-h}$, can be rewritten as

$$R_{i,j} = \sum_{k=1}^{h} \Big((1-\rho)\rho^{k-1}\Big) \circ I_{i,j-k} + \rho^{h} \circ R_{i,j-h} \tag{2.6}$$

depending on the given information of past experience. The closed claims with known number of unclosed claims $N_{i,j-h}$ in (2.5) can be interpreted as the summation of number of closed claims from all newly reported number of claims with the delay of $j-k$ years for $k = 1, 2, ..., h-1$ and the closed claims in development year $j-h$ from the total number of unclosed claims in the past. The probability of being closed for each unclosed claim in $N_{i,j-h}$ in development year j is the same as to the probability of being unclosed until development year $j-1$ and being closed in following year which is $(1-\rho)\rho^{h-1}$. The closed claims $R_{i,j}$ with given $R_{i,j-h}$ in (2.6) can be interpreted as the summation of the number of closed claims from all newly reported number of claims with the delay of $j-k$ years for $k = 1, 2, ..., h$, and the number of reopened claims from the closed claims which incur payments in development year $j-h$.

Proposition 2.3.2. The conditional probability function for closed number of claims $R_{i,j}$ is given by

$$\mathrm{P}(R_{i,j} = r_{i,j} \,|\, R_{i,j-1} = r_{i,j-1}) = \sum_{y=0}^{M_{i,j}} \sum_{i^*=r_{i,j}-y}^{\infty} c(r_{i,j}, y, i^*) \cdot R(r_{i,j}, y, i^*; \rho) \cdot \Lambda(i^*; \mu_i, \gamma_{j-1}) \tag{2.7}$$

where

$$\begin{cases} M_{i,j} = \min\{r_{i,j-1}, r_{i,j}\}, \\ c(r_{i,j}, y, i^*) = \binom{r_{i,j-1}}{y} \cdot \binom{i^*}{r_{i,j}-y}, \\ R(r_{i,j}, y, i^*; \rho) = \rho^{i^*-(r_{i,j}-2y)} \cdot (1-\rho)^{r_{i,j-1}+(r_{i,j}-2y)}, \\ \Lambda(i^*; \mu_i, \gamma_{j-1}) = \frac{e^{-\mu_i \gamma_{j-1}} \cdot (\mu_i \gamma_{j-1})^{i^*}}{i^*!}. \end{cases} \tag{2.8}$$

Proof. From (2.3), we see that random variable $R_{i,j}$ is a sum of two random variables. Then, the conditional probability function can be written as using convolution.

$$\begin{aligned}
\mathrm{P}(R_{i,j} = r_{i,j} \,|\, R_{i,j-1} = r_{i,j-1}) &= \mathrm{P}\left(\rho \circ R_{i,j-1} + (1-\rho) \circ I_{i,j-1} = r_{i,j} \,|\, R_{i,j-1} = r_{i,j-1}\right) \\
&= \mathrm{P}\left(\sum_{k=1}^{r_{i,j-1}} Y_k + \sum_{k=1}^{I_{i,j-1}} Z_k = r_{i,j}\right) \\
&= \sum_{y=0}^{M_{i,j}} \left[\mathrm{P}\left(\sum_{k=1}^{r_{i,j-1}} Y_k = y\right) \cdot \mathrm{P}\left(\sum_{k=1}^{I_{i,j-1}} Z_k = r_{i,j} - y\right)\right] \qquad (2.9)
\end{aligned}$$

where $M_{i,j} = \min\{r_{i,j-1}, r_{i,j}\}$.
According to Assumption 2.3.1, we have

$$\mathrm{P}\left(\sum_{k=1}^{r_{i,j-1}} Y_k = y\right) = \binom{r_{i,j-1}}{y} \cdot \rho^y \cdot (1-\rho)^{r_{i,j-1}-y} \qquad (2.10)$$

and

$$\begin{aligned}
&\mathrm{P}\left(\sum_{k=1}^{I_{i,j-1}} Z_k = r_{i,j} - y\right) \\
&= \sum_{i^*=r_{i,j}-y}^{\infty} \left[\mathrm{P}\left(\sum_{k=1}^{I_{i,j-1}} Z_k = r_{i,j} - y \,|\, I_{i,j-1} = i^*\right) \cdot \mathrm{P}\left(I_{i,j-1} = i^*\right)\right] \\
&= \sum_{i^*=r_{i,j}-y}^{\infty} \left[\binom{i^*}{r_{i,j}-y} \cdot (1-\rho)^{r_{i,j}-y} \cdot \rho^{i^*-(r_{i,j}-y)} \cdot \left(\frac{e^{-\mu_i\gamma_{j-1}} \cdot (\mu_i\gamma_{j-1})^{i^*}}{i^*!}\right)\right] \qquad (2.11)
\end{aligned}$$

By substituting (2.9) with (2.10) and (2.11) and after rearranging the parameters, the probability function for $R_{i,j}$ can be rewritten as

$$\mathrm{P}(R_{i,j} = r_{i,j} \,|\, R_{i,j-1} = r_{i,j-1}) = \sum_{y=0}^{M_{i,j}} \sum_{i^*=r_{i,j}-y}^{\infty} c(r_{i,j}, y, i^*) \cdot R(r_{i,j}, y, i^*; \rho) \cdot \Lambda(i^*; \mu_i, \gamma_{j-1})$$

where functions c, R and Λ are given by (2.8). □

Note that there is no closed claims in the first development year (i.e. $j = 0$) by the assumption; thus, $M_{i,1}$ is always zero. We can treat this as a special case, and rewrite the unconditional distribution of $R_{i,1}$ as

$$\mathrm{P}(R_{i,1} = r_{i,1}) = \sum_{i^*=r_{i,1}}^{\infty} c(r_{i,1}, 0, i^*) \cdot R(r_{i,1}, 0, i^*; \rho) \cdot \Lambda(i^*; \mu_i, \gamma_0)$$

where

$$\begin{cases} c(r_{i,1}, y, i^*) = \binom{i^*}{r_{i,1}}, \\ R(r_{i,1}, y, i^*; \rho) = \rho^{i^*-r_{i,1}} \cdot (1-\rho)^{r_{i,1}}, \\ \Lambda(i^*; \mu_i, \gamma_0) = \frac{e^{-\mu_i \gamma_0} \cdot (\mu_i \gamma_0)^{i^*}}{i^*!}. \end{cases}$$

Proposition 2.3.3. The unconditional mean, variance and auto-covariance of $R_{i,j}$'s are given by

$$\mathrm{E}\left[R_{i,j}\right] = \mu_i \sum_{k=1}^{j} (1-\rho)\rho^{k-1} \cdot \gamma_{j-k}, \tag{2.12}$$

$$\mathrm{Var}\left[R_{i,j}\right] = \mu_i \sum_{k=1}^{j} (1-\rho)\rho^{k-1} \cdot \gamma_{j-k}, \tag{2.13}$$

$$\mathrm{Cov}\left[R_{i,j}, R_{i,j-h}\right] = (1-\rho)\rho^{h} \cdot \mu_i \sum_{k=1}^{j-h} \rho^{k-1} \cdot \gamma_{j-k-h}. \tag{2.14}$$

Proof. Since $\left((1-\rho)\rho^{k-1}\right) \circ I_{i,j-k}$ follows a binomial distribution, conditioning on that $I_{i,j-k}$ is known, with parameters $I_{i,j-k}$ and $(1-\rho)\rho^{k-1}$, we have the following conditional mean and variance:

$$\mathrm{E}\left[\left((1-\rho)\rho^{k-1}\right) \circ I_{i,j-k} \mid I_{i,j-k}\right] = (1-\rho)\rho^{k-1} \cdot I_{i,j-k}, \tag{2.15}$$

$$\mathrm{Var}\left[\left((1-\rho)\rho^{k-1}\right) \circ I_{i,j-k} \mid I_{i,j-k}\right] = (1-\rho)\rho^{k-1} \left(1 - (1-\rho) \cdot \rho^{k-1}\right) \cdot I_{i,j-k}. \tag{2.16}$$

By taking the expectation on both sides of (2.4) and using (2.15), we have

$$\begin{aligned} \mathrm{E}\left[R_{i,j}\right] &= \mathrm{E}\left[\sum_{k=1}^{j} \left((1-\rho)\rho^{k-1}\right) \circ I_{i,j-k}\right] \\ &= \sum_{k=1}^{j} \mathrm{E}\left[\mathrm{E}\left[\left((1-\rho)\rho^{k-1}\right) \circ I_{i,j-k} \mid I_{i,j-k}\right]\right] \\ &= \sum_{k=1}^{j} \mathrm{E}\left[(1-\rho)\rho^{k-1} \cdot I_{i,j-k}\right] \\ &= \sum_{k=1}^{j} (1-\rho)\rho^{k-1} \cdot \mu_i \gamma_{j-k}, \end{aligned}$$

which proves (2.12).

To prove variance expression (2.13), by using (2.15) and (2.16) we first get

$$\begin{aligned} &\mathrm{Var}\left[\left((1-\rho)\rho^{k-1}\right) \circ I_{i,j-k}\right] \\ &\quad = \mathrm{Var}\left[\mathrm{E}\left[\left((1-\rho)\rho^{k-1}\right) \circ I_{i,j-k} \mid I_{i,j-k}\right]\right] + \mathrm{E}\left[\mathrm{Var}\left[\left((1-\rho)\rho^{k-1}\right) \circ I_{i,j-k} \mid I_{i,j-k}\right]\right] \end{aligned}$$

$$
\begin{aligned}
&= \Big((1-\rho)\rho^{k-1}\Big)^2 \operatorname{Var}\left[I_{i,j-k}\right] + (1-\rho)\rho^{k-1} \cdot \Big(1-(1-\rho)\cdot\rho^{k-1}\Big) \operatorname{E}\left[I_{i,j-k}\right] \\
&= \Big((1-\rho)\rho^{k-1}\Big)^2 \cdot \mu_i\gamma_{j-k} + (1-\rho)\rho^{k-1} \cdot \Big(1-(1-\rho)\rho^{k-1}\Big) \cdot \mu_i\gamma_{j-k} \\
&= \mu_i(1-\rho)\rho^{k-1} \cdot \gamma_{j-k}.
\end{aligned}
$$

Since $I_{i,j}$'s are independently distributed for any $0 \leq i, j \leq I$, $\Big((1-\rho)\rho^{k-1}\Big) \circ I_{i,j-k}$, for $k = 1, 2, ..., j$ are also independent from each other. Then the variance expression of $R_{i,j}$ (2.13) follows immediately.

For the covariance of $R_{i,j}$ and $R_{i,j-h}$ for $h \geq 1$, we first rewrite $R_{i,j}$ and $R_{i,j-h}$, similar to (2.5) and (2.4), respectively, as

$$
R_{i,j} = \sum_{k=1}^{h} \Big((1-\rho)\rho^{k-1}\Big) \circ I_{i,j-k} + \Big((1-\rho)\rho^{h}\Big) \circ N_{i,j-h-1}, \tag{2.17}
$$

$$
\begin{aligned}
R_{i,j-h} &= \sum_{k=1}^{j-h} \Big((1-\rho)\rho^{k-1}\Big) \circ I_{i,j-h-k} \\
&= \sum_{k^*=h+1}^{j} \Big((1-\rho)\rho^{k^*-h-1}\Big) \circ I_{i,j-k^*}.
\end{aligned} \tag{2.18}
$$

As $I_{i,j}$'s are assumed to be independently Poisson distributed for $0 \leq i, j \leq I$, noting (2.17) and (2.18) we get

$$
\begin{aligned}
\operatorname{Cov}[R_{i,j}, R_{i,j-h}] &= \operatorname{Cov}\left[\sum_{k=1}^{h-1} \Big((1-\rho)\rho^{k-1}\Big) \circ I_{i,j-k} + \Big((1-\rho)\rho^{h}\Big) \circ N_{i,j-h-1}, R_{i,j-h}\right] \\
&= \operatorname{Cov}\left[\Big((1-\rho)\rho^{h}\Big) \circ N_{i,j-h-1}, R_{i,j-h}\right] \\
&= \operatorname{Cov}\left[\rho^{h} \circ R_{i,j-h}, R_{i,j-h}\right] \\
&= \operatorname{E}\left[\operatorname{Cov}\left[\rho^{h} \circ R_{i,j-h}, R_{i,j-h} \middle| R_{i,j-h}\right]\right] \\
&\quad + \operatorname{Cov}\left[\operatorname{E}\left[\rho^{h} \circ R_{i,j-h} \middle| R_{i,j-h}\right], \operatorname{E}\left[R_{i,j-h} \middle| R_{i,j-h}\right]\right] \\
&= 0 + \operatorname{Cov}\left[\rho^{h} \cdot R_{i,j-h}, R_{i,j-h}\right] \\
&= \operatorname{Cov}\left[\rho^{h} \cdot R_{i,j-h}, R_{i,j-h}\right] \\
&= \rho^{h} \cdot \operatorname{Var}[R_{i,j-h}]
\end{aligned}
$$

and then (2.14) follows by using expression (2.13). □

From Proposition 2.3.3, we have that the unconditional expectation of $R_{i,j}$ is the same as the unconditional variance. This draws conclusion that the Poisson INAR(1) model is a non-dispersed model which is not a desired property to have for claim counts model. In addition, we can get the conditional expectation and variance of $R_{i,j}$, given $R_{i,j-h}$, as follows:

$$\mathrm{E}\left[R_{i,j}\,|\,R_{i,j-h}=r_{i,j-h}\right]=(1-\rho)\left(\sum_{k=1}^{h}\rho^{k-1}\cdot\gamma_{j-k}\right)\mu_i+\rho^h r_{i,j-h}, \tag{2.19}$$

$$\mathrm{Var}\left[R_{i,j}\,|\,R_{i,j-h}=r_{i,j-h}\right]=(1-\rho)\left(\sum_{k=1}^{h}\rho^{k-1}\cdot\gamma_{j-k}\right)\mu_i+\rho^h(1-\rho^h)\cdot r_{i,j-h} \tag{2.20}$$

2.4 Individual Claim Size

There are often payments with the amount of zero, so-called "zero-claims" in non-life insurance; thus, we allow the presence of zero-claims for the individual claim size model. For claim size distribution, Weibull, Exponential, Gamma, and Pareto distribution can be adopted depending on the characteristics of the data observed. However, inspired by Tweedie's compound Poisson model (see also Margraf 2017), we focus on Gamma distribution for claim sizes in this project.

2.4.1 Model Assumptions

Let $X_{i,j}^{(l)}$ be the individual claim size of the payment for l^{th} claim in accident year i and development year j. We assume that individual claim size $X_{i,j}^{(l)}$ follows a zero adjusted gamma distribution (ZAGA) (or mixed gamma distribution) with parameter $q_{i,j}$ being the probability of zero-claims that is $\mathrm{P}\left(X_{i,j}^{(l)}=0\right)=q_{i,j}$ to allow the presence of zero-claims.

Assumption 2.4.1.

- Conditioning on that the claim amount paid is non-zero, the individual claim sizes (payments) follow a gamma distribution with mean $\tau_{i,j}>0$ and shape parameter $\alpha>0$ that is,

$$X_{i,j}^{(l)}\,|\,X_{i,j}^{(l)}>0\sim gamma\left(\alpha,\frac{\alpha}{\tau_{i,j}}\right),\quad l=1,2,....$$

  Denote its density function as $f_{X_{i,j}|X_{i,j}>0}(x;\alpha,\tau_{i,j})$.

- The sequence of individual sizes $\{X_{i,j}^{(l)},l\in\mathbb{N}^+\}$ are independent and identically distributed for any fixed i and j, $0\le i,j\le I$ with a mixed gamma distribution; its density function can be expressed as

$$f_{X_{i,j}}(x;q_{i,j},\alpha,\tau_{i,j})=\begin{cases}q_{i,j}, & \text{if } x=0\\ (1-q_{i,j})\cdot f_{X_{i,j}|X_{i,j}>0}(x;\alpha,\tau_{i,j}), & \text{if } x>0\end{cases}$$

$$= q_{i,j} \cdot 1_{\{x=0\}} + (1 - q_{i,j}) \cdot \frac{\left(\frac{\alpha}{\tau_{i,j}}\right)^{\alpha}}{\Gamma(\alpha)} (x)^{\alpha-1} e^{-\frac{\alpha}{\tau_{i,j}} x}, \qquad x \geq 0. \tag{2.21}$$

Remark 2.4.1.

- Although it is reasonable to assume that the probability of zero payments $q_{i,j}$ depends on both the accident year and development year, in this project, we assume that the probability of zero claims only depends on development year j for the simplicity for maximum likelihood estimations in next chapter.

- In the paper proposed by Margraf (2017), given that the individual payments are non-zero, the conditional mean $\mu_{i,j}$ and the conditional variance $\sigma_{i,j}$ are assumed to be depending on accident year i and development year j. Thus, new parameters for the different inflation rates γ_i and δ_i are introduced, and the conditional mean and the conditional variance are denoted as $\mu_{i,j} = \mu\gamma_i\delta_j$ and $\sigma^2_{i,j} = \sigma^2\gamma_i^2\delta_j^2$.

- Generally, the mean of total claim size in each cell (i, j) may be different for different values of i and j. Further, it is reasonable to have two different factors that impact the mean of total claim size for accident year i and development year j. However, in this project, we assume that there is only a constant inflation δ, which is known from the past experience, impacting the claim size for different i and j, and define the individual claim size in cell (i, j) as

$$X^{(l)}_{i,j} \overset{d}{=} X^{(l)}_{0,1} \cdot \delta^{i+j-1}. \tag{2.22}$$

  where, $X^{(l)}_{0,1}$ follows a mixed gamma distribution with mean $\tau > 0$ and shape parameter $\alpha > 0$ with a mass probability at zero q_j that is $\mathrm{P}\left(X^{(l)}_{0,1} = 0\right) = q_j$. Since we assume that the claims are not closed within the same year, there is no closed claim when $j = 0$. Thus, a constant inflation is applied to $X^{(l)}_{0,1}$, the size of claims in first calender year, depends on different calender year $i + j$. This model for claim sizes is used in later chapter for estimating model parameters and simulation studies.

2.4.2 Properties

The proposed zero adjusted gamma model for individual claim size has following properties.

Proposition 2.4.1. The mean and variance of $X^{(l)}_{i,j}$ are obtained as

$$\mathrm{E}\left[X^{(l)}_{i,j}\right] = (1 - q_{i,j}) \cdot \tau_{i,j}, \tag{2.23}$$

$$\mathrm{Var}\left[X^{(l)}_{i,j}\right] = (1 - q_{i,j}) \cdot \tau^2_{i,j} \left(\frac{1}{\alpha} + q_{i,j}\right). \tag{2.24}$$

As first and second moment are zero conditioning on $X_{i,j}^{(l)} = 0$, the first and second moment can be explicitly derived by multiplying the probability of non-zero claims and integration of gamma density function.

2.5 Incremental Aggregate Claims

Recall that $R_{i,j}$ is the number of closed claims incurred in accident year i and payments are made in development year j, and $X_{i,j}^{(l)}$ is the size of the l^{th} claim payment which have been studied in Sections 2.3 and 2.4, respectively. In this section, we combine these two models to study the total amount of payments for those claims incurred in accident year i and payments are made in development year j. This latter random variable is called in this project as the incremental aggregate claims in cell (i, j), and denoted by $S_{i,j}$. Clearly, we have

$$S_{i,j} = \sum_{l=1}^{R_{i,j}} X_{i,j}^{(l)}.$$

It is the total (aggregate) incremental paid payments for claims that occur in accident year i and have been “closed" in development year j where these closed claims may be re-opened and then re-closed in the future development years.

2.5.1 Model Assumptions

As we have discussed in previous sections, we use a Poisson INAR(1) model for the number of closed claims $R_{i,j}$. For the sizes of claims $X_{i,j}^{(l)}$, a zero adjusted gamma distribution is used to model the individual claim payments for each cell of (i, j). We present below the assumptions for the incremental aggregate claims model.

Assumption 2.5.1.

- Closed number of claims $R_{i,j}$ of different accident years i are independent, and it follows an INAR(1) process such that

$$R_{i,j} = \rho \circ R_{i,j-1} + (1-\rho) \circ I_{i,j-1}, \qquad 0 \le i, j \le I,$$

  with $R_{i,0} = 0$.

- Assumption 2.3.1 is applied

- Th individual claim sizes $X_{i,j}^{(l)}$, $l \in \mathbb{N}^+$ are i.i.d. random variables for each cell (i, j) and $X_{i,j}^{(1)}$ follows a mixed gamma distribution with mean $\tau_{i,j} > 0$, shape parameter $\alpha > 0$ and a mass probability of $q_{i,j}$ at zero.

- $R_{i,j}$ and $X_{i,j}^{(l)}$'s are independent for any i and j.

Remark 2.5.1.

- The assumption of independence between $R_{i,j}$ and $X_{i,j}^{(l)}$ may not be valid in practice. For claims with longer development year, the number of closed claim is likely to be decreased where as the amount of payments are larger for the same reason as mentioned in Remark 2.4.1. However, the assumption is made for the simplicity, and we believe that the model with this assumption provides the useful insights for the future investigations.

2.5.2 Properties

The proposed model for incremental aggregate claim have following properties.

Proposition 2.5.1. The mean and variance of $S_{i,j}$ are obtained as

$$\mathrm{E}\left[S_{i,j}\right] = (1-q_{i,j}) \cdot \left(\sum_{k=1}^{j}(1-\rho)\rho^{k-1} \cdot \gamma_{j-k}\right) \cdot \mu_i \cdot \tau_{i,j} \tag{2.25}$$

$$\mathrm{Var}\left[S_{i,j}\right], = (1-q_{i,j}) \cdot \left(1+\frac{1}{\alpha}\right) \cdot \left(\sum_{k=1}^{j}(1-\rho)\rho^{k-1} \cdot \gamma_{j-k}\right) \cdot \mu_i \cdot \tau_{i,j}^2.$$

Proof. Since $X_{i,j}^{(l)}$, $l \in \mathbb{N}^+$ are independent and zero adjusted gamma distributed with $\mathrm{P}\left(X_{i,j}^{(l)}=0\right) = q_{i,j}$, the conditional mean and variance of $S_{i,j}$ given $R_{i,j}$ can be obtained as

$$\mathrm{E}\left[S_{i,j} \mid R_{i,j}\right] = \mathrm{E}\left[\sum_{l=1}^{R_{i,j}} X_{i,j}^{(l)} \mid R_{i,j}\right] = R_{i,j} \cdot \tau_{i,j}(1-q_{i,j}) \tag{2.26}$$

$$\mathrm{Var}\left[S_{i,j} \mid R_{i,j}\right] = \mathrm{Var}\left[\sum_{l=1}^{R_{i,j}} X_{i,j}^{(l)} \mid R_{i,j}\right] = R_{i,j} \cdot (1-q_{i,j})\tau_{i,j}^2\left(\frac{1}{\alpha}+q_{i,j}\right) \tag{2.27}$$

By taking the expectation on both sides of (2.26) and (2.27) and using the results given in Propositions 2.4.1 and 2.3.3, we have

$$\begin{aligned}
\mathrm{E}\left[S_{i,j}\right] &= \mathrm{E}\left[\mathrm{E}\left[\sum_{l=1}^{R_{i,j}} X_{i,j}^{(l)} \mid R_{i,j}\right]\right] \\
&= \mathrm{E}\left[R_{i,j} \cdot (1-q_{i,j}) \cdot \tau_{i,j}\right] \\
&= (1-q_{i,j})\tau_{i,j} \cdot \mathrm{E}\left[R_{i,j}\right] \\
&= (1-q_{i,j})\tau_{i,j}\left(\sum_{k=1}^{j}(1-\rho) \cdot \rho^{k-1} \cdot \gamma_{j-k}\right) \cdot \mu_i
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Var}\left[S_{i,j}\right] &= \mathrm{Var}\left[\sum_{k=1}^{R_{i,j}} X_{i,j}^{(l)}\right] \\
&= \mathrm{Var}\left[\mathrm{E}\left[\sum_{k=1}^{R_{i,j}} X_{i,j}^{(l)} \mid R_{i,j}\right]\right] + \mathrm{E}\left[\mathrm{Var}\left[\sum_{k=1}^{R_{i,j}} X_{i,j}^{(l)} \mid R_{i,j}\right]\right] \\
&= \mathrm{Var}\left[R_{i,j} \cdot (1-q_{i,j})\tau_{i,j}\right] + \mathrm{E}\left[R_{i,j} \cdot (1-q_{i,j})\tau_{i,j}^2 \left(\frac{1}{\alpha} + q_{i,j}\right)\right] \\
&= (1-q_{i,j})^2 \cdot \tau_{i,j}^2 \cdot \mathrm{Var}\left[R_{i,j}\right] + (1-q_{i,j}) \cdot \tau_{i,j}^2 \left(\frac{1}{\alpha} + q_{i,j}\right) \cdot \mathrm{E}\left[R_{i,j}\right] \\
&= (1-q_{i,j})\tau_{i,j}^2 \left(1 - q_{i,j} + \frac{1}{\alpha} + q_{i,j}\right) \cdot \mathrm{E}\left[R_{i,j}\right] \\
&= (1-q_{i,j})\tau_{i,j}^2 \left(1 + \frac{1}{\alpha}\right) \cdot \left(\sum_{k=1}^{j} (1-\rho) \cdot \rho^{k-1} \cdot \gamma_{j-k}\right) \mu_i.
\end{aligned}$$

□

According to (2.25), variance of $S_{i,j}$ can be rewritten as a function of an expectation as

$$\mathrm{Var}\left[S_{i,j}\right], = \tau_{i,j} \cdot \left(1 + \frac{1}{\alpha}\right) \cdot \mathrm{E}\left[S_{i,j}\right].$$

In the case that the mean of individual claims $\tau_{i,j}$ is more than 1, and $\alpha > 0$ by the assumption. Therefore, $\tau_{i,j} \cdot \left(1 + \frac{1}{\alpha}\right) > 1$, and it is concluded that the compound model based on the Poisson INAR(1) model for claims counts with a mixed gamma distribution for the size is an over-dispersed model which is one of the desired properties for the model for incremental aggregate claims.

Chapter 3

Estimation of Model Parameters

The replicated time series model of INAR of order one has been studied in Silva (2005). From all the parameter estimation methods discussed in there, we adopt the maximum likelihood estimation method for estimating parameters of Poisson INAR(1) model for claim counts $R_{i,j}$ and compound INAR(1) model for incremental aggregate claims $S_{i,j}$. In Section 3.1, further assumptions mentioned in Remark 2.4.1 are introduced. Sections 3.3 and 3.4 present the estimation of parameters of these models using MLE technique.

3.1 Additional Assumptions and Simplifications

As we have mentioned in Remark 2.4.1, a special severity model is to be used in this chapter for illustrating the MLE technique. In this model, the probability of zero-claims depends only on j and the constant inflation is an only factor having an impact on individual claim sizes in different accident years as well as different development years. Using (2.21), we can write the probability density function of $X_{i,j}^{(l)}$ as

$$\begin{aligned} f_{X_{i,j}}(x; q_j, \alpha, \tau, \delta) &= \begin{cases} q_j, & \text{if } x = 0 \\ (1-q_j) \cdot f_{X_{i,j}^{(l)} | X_{i,j}^{(l)} > 0}(x; \alpha, \tau, \delta), & \text{if } x > 0 \end{cases} \\ &= q_j \cdot 1_{\{x=0\}} + (1-q_j) \cdot \frac{\left(\frac{\alpha}{\tau\delta^{i+j-1}}\right)^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{\alpha}{\tau\delta^{i+j-1}} x}, \qquad x \geq 0, \end{aligned} \tag{3.1}$$

where we assume that $X_{0,1}^{(l)} \mid X_{0,1}^{(l)} > 0 \sim gamma(\alpha, \frac{\alpha}{\tau})$, that is,

$$f_{X_{i,j}|X_{i,j}>0}(x; \alpha, \tau_{i,j}) = \frac{\left(\frac{\alpha}{\tau_{i,j}}\right)^{\alpha}}{\Gamma(\alpha)} (x)^{\alpha-1} e^{-\frac{\alpha}{\tau_{i,j}} x}, \qquad x > 0.$$

By our relationship assumption (2.22), when $X_{i,j}^{(l)} > 0$, we have

$$\begin{aligned} F_{X_{i,j}}(x;\alpha,\tau) &= \mathrm{P}\left(X_{i,j}^{(l)} < x\right) \\ &= \mathrm{P}\left(X_{0,1}^{(l)} \cdot \delta^{i+j-1} < x\right) \\ &= \mathrm{P}\left(X_{0,1}^{(l)} < \frac{x}{\delta^{i+j-1}}\right), \end{aligned}$$

and then by taking the derivative on both side w.r.t. x, we have

$$\begin{aligned} f_{X_{i,j}}(x;\alpha,\tau) &= \frac{d}{dx}\mathrm{P}\left(X_{0,1}^{(l)} < \frac{x}{\delta^{i+j-1}}\right) \\ &= \frac{1}{\delta^{i+j-1}} f_{X_{0,1}^{(l)}}\left(\frac{x}{\delta^{i+j-1}};\alpha,\tau\right) \\ &= \frac{\left(\frac{\alpha}{\tau\delta^{i+j-1}}\right)^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\frac{\alpha}{\tau\delta^{i+j-1}}x}. \end{aligned}$$

From (3.1), we can notice that $X_{i,j}^{(l)} | X_{i,j}^{(l)} > 0$ has gamma distribution with shape parameter α and scale parameter $\frac{\alpha}{\tau\delta^{i+j-1}}$. Thus, unconditional mean and variance of $X_{i,j}^{(l)}$ can be easily obtained as

$$\mathrm{E}\left[X_{i,j}^{(l)}\right] = (1-q_j)\cdot\tau\delta^{i+j-1}, \tag{3.2}$$

$$\mathrm{Var}\left[X_{i,j}^{(l)}\right] = (1-q_j)\cdot(\tau\delta^{i+j-1})^2\left(\frac{1}{\alpha}+q_j\right). \tag{3.3}$$

by replacing $\tau_{i,j}$ and $q_{i,j}$ with $\tau\delta^{i+j-1}$ and q_j respectively in (2.23) and (2.24).

In this project, the constant inflation is a known factor from past experience. The inflation rate can be treated as an unknown factor and non-constant; it can be studied and modeled separately using past data for inflation rates to forecast future inflation.

3.2 Data and Notation

For simplicity, we denote by $\mathcal{A}_I = \{(i,j) \in \mathbb{N}_0 \times \mathbb{N} : 0 \le i+j \le I\}$. Then we can define the number of closed claims that are observed as $D_I = \{R_{i,j} :\ (i,j) \in \mathcal{A}_I\}$, and all the amounts of individual claim payments paid by the insurer as $\Delta_I = \{X_{i,j}^{(l)} :\ 1 \le l \le R_{i,j},\ (i,j) \in \mathcal{A}_I\}$. For notation simplicity, we define the following sets of parameters in our model:

$$\begin{aligned} \boldsymbol{\mu} &= \{\mu_0, \mu_1, ..., \mu_{I-1}\}, \\ \boldsymbol{\gamma} &= \{\gamma_0, \gamma_1, ..., \gamma_{I-1}\}, \\ \boldsymbol{q} &= \{q_1, q_2, ..., q_I\}, \end{aligned}$$

and further define the following

$$\boldsymbol{\theta} = \{\rho, \boldsymbol{\mu}, \boldsymbol{\gamma}, \boldsymbol{q}, \alpha, \tau\},$$

as the set of parameters that need to be estimated.

As we assume that $R_{i,j}$ and $X_{i,j}^{(l)}$'s are independent, we can write the likelihood function as

$$L_{D_I,\Delta_I}(\boldsymbol{\theta}) = L_{D_I}(\rho, \boldsymbol{\mu}, \boldsymbol{\gamma}) \times L_{\Delta_I|D_I}(\boldsymbol{q}, \alpha, \tau).$$

Since L_{D_I} and $L_{\Delta_I|D_I}$ are functions of different parameters, we can study maximum likelihood functions separately to maximize L_{D_I,Δ_I} (see, for example, Verrall et al., 2010 and Gao et al., 2013).

3.3 Maximum Likelihood Estimation of Count parameters

The likelihood function of all observed counts $\{R_{i,j};\ (i,j) \in \mathcal{A}_I\}$ is given by

$$\begin{aligned} L_{D_I}(\rho, \boldsymbol{\mu}, \boldsymbol{\gamma}) &= \prod_{i=0}^{I-1} \left[\mathrm{P}(R_{i,1} = r_{i,1}) \prod_{j=2}^{I-i} \mathrm{P}(R_{i,j} = r_{i,j} \mid R_{i,j-1} = r_{i,j-1}) \right] \\ &= \left(\prod_{i=0}^{I-1} \mathrm{P}(R_{i,1} = r_{i,1}) \right) \left(\prod_{i=0}^{I-1} \prod_{j=2}^{I-i} \mathrm{P}(R_{i,j} = r_{i,j} \mid R_{i,j-1} = r_{i,j-1}) \right). \end{aligned} \tag{3.4}$$

Using expression (2.7), the corresponding log-likelihood function can be easily obtained by taking logarithms on both sides of (3.4) as

$$\begin{aligned} l_{D_I}(\rho, \boldsymbol{\mu}, \boldsymbol{\gamma}) &= \sum_{i=0}^{I-1} \log(\mathrm{P}(R_{i,1} = r_{i,1})) + \sum_{i=0}^{I-1} \sum_{j=2}^{I-i} \log(\mathrm{P}(R_{i,j} = r_{i,j} \mid R_{i,j-1} = r_{i,j-1})) \qquad (3.5) \\ &= \sum_{i=0}^{I-1} \sum_{j=1}^{I-i} \log \left(\sum_{y=0}^{M_{i,j}} \sum_{i^*=r_{i,j}-y}^{\infty} c(r_{i,j}, y, i^*) \cdot R(r_{i,j}, y, i^*; \rho) \cdot \Lambda(i^*; \mu_i, \gamma_{j-1}) \right), \end{aligned}$$

where $M_{i,j} = \min\{r_{i,j-1}, r_{i,j}\}$ and functions c, R and Λ are given by (2.8).

Since c, R and Λ are functions of different parameters, we can easily get partial derivatives of the log-likelihood function (3.5) with respect to (w.r.t.) parameters ρ, μ_i and γ_{j-1}, respectively. By setting each of these partial derivatives equal to zero, we can get maximum likelihood estimating equation for ρ, μ_i's and γ_{j-1}'s for $0 \le i \le I-1$, $1 \le j \le I$, and we state the results in the theorem below.

Theorem 3.3.1. The maximum likelihood estimations of parameters ρ, μ_i's and γ_j's satisfy the following system of estimating equations:

$$\rho = \frac{\sum_{i=0}^{I-1}\left(\mu_i \sum_{j=1}^{I-i}\gamma_{j-1} + 2\sum_{j=2}^{I-i}\sum_{y=0}^{M_{i,j}}\sum_{i^*=r_{i,j}-y}^{\infty} H_{i,j}(y,i^*;\rho,\mu_i,\gamma_{j-1})\cdot y\right) - r_{\bullet,\bullet}}{\sum_{i=0}^{I-1}\sum_{j=1}^{I-i}(\mu_i\gamma_{j-1} + r_{i,j-1})} \tag{3.6}$$

$$\mu_i = \frac{\sum_{j=1}^{I-i}\sum_{y=0}^{M_{i,j}}\sum_{i^*=r_{i,j}-y}^{\infty} H_{i,j}(y,i^*;\rho,\mu_i,\gamma_{j-1})\cdot i^*}{\sum_{j=1}^{I-i}\gamma_{j-1}}, \qquad i = 0,1,...,I-1, \tag{3.7}$$

$$\gamma_{j-1} = \frac{\sum_{i=0}^{I-j}\sum_{y=0}^{M_{i,j}}\sum_{i^*=r_{i,j}-y}^{\infty} H_{i,j}(y,i^*;\rho,\mu_i,\gamma_{j-1})\cdot i^*}{\sum_{i=0}^{I-j}\mu_i}, \qquad j = 1,2,...,I, \tag{3.8}$$

where function $H_{i,j}$ is defined as

$$H_{i,j}(y,i^*;\rho,\mu_i,\gamma_{j-1}) = \frac{c(r_{i,j},y,i^*)\cdot R(r_{i,j},y,i^*;\rho)\cdot \Lambda(i^*;\mu_i,\gamma_{j-1})}{\mathrm{P}\left(R_{i,j}=r_{i,j}\,|\,R_{i,j-1}=r_{i,j-1}\right)}, \tag{3.9}$$
$$0 \le i \le I-1,\ 1 \le j \le I,\ 0 \le y \le M_{i,j} \text{ and } r_{i,j}-y \le i^* < \infty,$$

in which the expression of conditional probability $\mathrm{P}\left(R_{i,j}=r_{i,j}\,|\,R_{i,j-1}=r_{i,j-1}\right)$ is given by (2.7) and functions c, R and Λ are showed in (2.8), and

$$r_{\bullet,\bullet} = \sum_{i=0}^{I-1}\sum_{j=1}^{I-i} r_{i,j},$$

which is the total observed number of closed claims in the upper left triangle. We also note that for any function f, $\sum_{i=a}^{b} f(i) = 0$ if $a > b$.

Proof. First, we find the partial derivatives of $R(y,i^*;\rho)$ w.r.t. ρ, and $\Lambda(i^*;\mu_i,\gamma_{j-1})$ w.r.t. μ_i and γ_{j-1} as follows:

$$\begin{aligned}\frac{\partial}{\partial\rho}R(y,i^*;\rho) &= \frac{\partial}{\partial\rho}\left(\rho^{i^*-(r_{i,j}-2y)}\cdot(1-\rho)^{r_{i,j-1}+(r_{i,j}-2y)}\right)\\ &= \rho^{i^*-(r_{i,j}-2y)}\cdot(1-\rho)^{r_{i,j-1}+(r_1-2y)}\cdot\frac{i^*-r_{i,j}+2y-i^*\cdot\rho-r_{i,j-1}\cdot\rho}{\rho\cdot(1-\rho)}\\ &= R(y,i^*;\rho)\cdot\frac{i^*(1-\rho)-r_{i,j}+2y-r_{i,j-1}\cdot\rho}{\rho\cdot(1-\rho)}, \qquad y\ge 0 \end{aligned} \tag{3.10}$$

$$\frac{\partial}{\partial\mu_i}\Lambda(i^*;\mu_i,\gamma_{j-1}) = \frac{\partial}{\partial\mu_i}\left(\frac{e^{-\mu_i\gamma_{j-1}}\cdot(\mu_i\gamma_{j-1})^{i^*}}{i^*!}\right)$$

$$= (-\gamma_{j-1}) \frac{e^{-\mu_i \gamma_{j-1}} \cdot (\mu_i \gamma_{j-1})^{i^*}}{i^*!} + \frac{e^{-\mu_i \gamma_{j-1}} \cdot (\mu_i \gamma_{j-1})^{i^*-1}}{(i^*-1)!} \cdot \gamma_{j-1}$$

$$= \Lambda(i^*; \mu_i, \gamma_{j-1}) \cdot \left(\frac{i^*}{\mu_i} - \gamma_{j-1} \right), \tag{3.11}$$

$$\frac{\partial}{\partial \gamma_{j-1}} \Lambda(i^*; \mu_i, \gamma_{j-1}) = \frac{\partial}{\partial \gamma_{j-1}} \left(\frac{e^{-\mu_i \gamma_{j-1}} \cdot (\mu_i \gamma_{j-1})^{i^*}}{i^*!} \right)$$

$$= \Lambda(i^*; \mu_i, \gamma_{j-1}) \cdot \left(\frac{i^*}{\gamma_{j-1}} - \mu_i \right). \tag{3.12}$$

From (3.10)-(3.12), we can also get

$$\begin{aligned} &\frac{\partial}{\partial \rho} \mathrm{P}\left(R_{i,j} = r_{i,j} \,|\, R_{i,j-1} = r_{i,j-1}\right) \\ &\quad = \sum_{y=0}^{M_{i,j}} \sum_{i^*=r_{i,j}-y}^{\infty} c(y, i^*) \cdot \left(\frac{\partial}{\partial \rho} R(y, i^*; \rho) \right) \cdot \Lambda(i^*, \mu_i, \gamma_{j-1}) \\ &\quad = \sum_{y=0}^{M_{i,j}} \sum_{i^*=r_{i,j}-y}^{\infty} c(y, i^*) \cdot R(y, i^*; \rho) \cdot \Lambda(i^*, \mu_i, \gamma_{j-1}) \cdot \frac{i^*(1-\rho) - r_{i,j} + 2y - r_{i,j-1} \cdot \rho}{\rho \cdot (1-\rho)}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\frac{\partial}{\partial \mu_i} \mathrm{P}\left(R_{i,j} = r_{i,j} \,|\, R_{i,j-1} = r_{i,j-1}\right) \\ &\quad = \sum_{y=0}^{M_{i,j}} \sum_{i^*=r_{i,j}-y}^{\infty} c(y, i^*) \cdot R(y, i^*; \rho) \cdot \left(\frac{\partial}{\partial \mu_i} \Lambda(i^*; \mu_i, \gamma_{j-1}) \right) \\ &\quad = \sum_{y=0}^{M_{i,j}} \sum_{i^*=r_{i,j}-y}^{\infty} c(y, i^*) \cdot R(y, i^*; \rho) \cdot \Lambda(i^*; \mu_i, \gamma_{j-1}) \cdot \left(\frac{i^*}{\mu_i} - \gamma_{j-1} \right), \end{aligned} \tag{3.14}$$

$$\begin{aligned} &\frac{\partial}{\partial \gamma_{j-1}} \mathrm{P}\left(R_{i,j} = r_{i,j} \,|\, R_{i,j-1} = r_{i,j-1}\right) \\ &\quad = \sum_{y=0}^{M_{i,j}} \sum_{i^*=r_{i,j}-y}^{\infty} c(y, i^*) \cdot R(y, i^*; \rho) \cdot \Lambda(i^*; \mu_i, \gamma_{j-1}) \cdot \left(\frac{i^*}{\gamma_{j-1}} - \mu_i \right), \qquad j = 1, 2, ...I. \end{aligned} \tag{3.15}$$

Then taking the partial derivative of loglikelihood function (3.5) w.r.t. ρ, using (3.13), and noting the definition of $H_{i,j}$ given by (3.9) and

$$\sum_{y=1}^{M_{i,j}} \sum_{i^*=r_{i,j}-y}^{\infty} H_{i,j}(y, i^*; \rho, \mu_i, \gamma_{j-1}) = 1, \qquad \text{for } 0 \le i \le I-1, 1 \le j \le I,$$

give

$$\begin{aligned}
\frac{\partial l_{D_I}}{\partial \rho} &= \sum_{i=0}^{I-1} \frac{\frac{\partial}{\partial \rho}\mathrm{P}\left(R_{i,1}=r_{i,1}\right)}{\mathrm{P}\left(R_{i,1}=r_{i,1}\right)} + \sum_{i=0}^{I-1}\sum_{j=2}^{I-i} \frac{\frac{\partial}{\partial \rho}\mathrm{P}\left(R_{i,j}=r_{i,j} \,|\, R_{i,j-1}=r_{i,j-1}\right)}{\mathrm{P}\left(R_{i,j}=r_{i,j} \,|\, R_{i,j-1}=r_{i,j-1}\right)} \\
&= \frac{1}{\rho}\sum_{i=0}^{I-1}\sum_{j=1}^{I-i}\left[\sum_{y=0}^{M_{i,j}}\sum_{i^*=r_{i,j}-y}^{\infty} \frac{c(y,i^*)\cdot R(y,i^*;\rho)\cdot \Lambda(i^*;\mu_i,\gamma_{j-1})\cdot (i^*+\frac{2y}{1-\rho})}{\mathrm{P}\left(R_{i,j}=r_{i,j} \,|\, R_{i,j-1}=r_{i,j-1}\right)}\right. \\
&\quad \left. - \frac{(r_{i,j}+\rho\cdot r_{i,j-1})}{1-\rho}\sum_{y=0}^{M_{i,j}}\sum_{i^*=r_{i,j}-y}^{\infty} \frac{c(y,i^*)\cdot R(y,i^*;\rho)\cdot \Lambda(i^*;\mu_i,\gamma_{j-1})}{\mathrm{P}\left(R_{i,j}=r_{i,j} \,|\, R_{i,j-1}=r_{i,j-1}\right)}\right] \\
&= \frac{1}{\rho}\sum_{i=0}^{I-1}\sum_{j=1}^{I-i}\left(\sum_{y=0}^{M_{i,j}}\sum_{i^*=r_{i,j}-y}^{\infty} H_{i,j}(y,i^*;\rho,\mu_i,\gamma_{j-1})\cdot\left(i^*+\frac{2y}{1-\rho}\right) - \frac{r_{i,j}+\rho\cdot r_{i,j-1}}{1-\rho}\right) \\
&= \frac{1}{\rho}\sum_{i=0}^{I-1}\sum_{j=1}^{I-i}\sum_{y=0}^{M_{i,j}}\sum_{i^*=r_{i,j}-y}^{\infty} H_{i,j}(y,i^*;\rho,\mu_i,\gamma_{j-1})\cdot\left(i^*+\frac{2y}{1-\rho}\right) - \frac{\mathbf{r}_{\bullet,\bullet}+\rho\sum_{i=0}^{I-1}\sum_{j=1}^{I-i} r_{i,j-1}}{\rho(1-\rho)}.
\end{aligned} \tag{3.16}$$

Now, taking the partial derivative of (3.5) w.r.t. μ_i and using (3.14) give

$$\begin{aligned}
\frac{\partial l_{D_I}}{\partial \mu_i} &= \frac{\frac{\partial}{\partial \mu_i}\mathrm{P}\left(R_{i,1}=r_{i,1}\right)}{\mathrm{P}\left(R_{i,1}=r_{i,1}\right)} + \sum_{j=2}^{I-i} \frac{\frac{\partial}{\partial \mu_i}\mathrm{P}\left(R_{i,j}=r_{i,j} \,|\, R_{i,j-1}=r_{i,j-1}\right)}{\mathrm{P}\left(R_{i,j}=r_{i,j} \,|\, R_{i,j-1}=r_{i,j-1}\right)} \\
&= \frac{1}{\mu_i}\sum_{j=1}^{I-i}\sum_{y=0}^{M_{i,j}}\sum_{i^*=r_{i,j}-y}^{\infty} H_{i,j}(y,i^*;\rho,\mu_i,\gamma_{j-1})\cdot i^* - \sum_{j=1}^{I-i}\gamma_{j-1}.
\end{aligned} \tag{3.17}$$

Similarly, by taking the partial derivative of loglikelihood function (3.5) w.r.t. γ_{j-1} for $j = 1, 2, ..., I$, respectively, and using (3.15), we get

$$\begin{aligned}
\frac{\partial l_{D_I}}{\partial \gamma_{j-1}} &= \sum_{i=0}^{I-j} \frac{\frac{\partial}{\partial \gamma}\mathrm{P}\left(R_{i,j}=r_{i,j} \,|\, R_{i,j-1}=r_{i,j-1}\right)}{\mathrm{P}\left(R_{i,j}=r_{i,j} \,|\, R_{i,j-1}=r_{i,j-1}\right)} \\
&= \sum_{i=0}^{I-j}\sum_{y=0}^{M_{i,j}}\sum_{i^*=r_{i,j}-y}^{\infty} \frac{c(y,i^*)\cdot R(y,i^*;\rho)\cdot \Lambda(i^*;\mu_i,\gamma_{j-1})\cdot\left(\frac{i^*}{\gamma_{j-1}}-\mu_i\right)}{\mathrm{P}\left(R_{i,j}=r_{i,j} \,|\, R_{i,j-1}=r_{i,j-1}\right)} \\
&= \frac{1}{\gamma_{j-1}}\sum_{i=0}^{I-1}\sum_{y=0}^{M_{i,j}}\sum_{i^*=r_{i,j}-y}^{\infty} H_{i,j}(y,i^*;\rho,\mu_i,\gamma_{j-1})\cdot i^* - \sum_{i=0}^{I-1}\mu_i.
\end{aligned} \tag{3.18}$$

Finally, estimating equations (3.7) and (3.8) can be easily obtained by letting (3.17) and (3.18) be zero, and estimating equation (3.6) can be obtained by setting (3.16) to be zero and using equation (3.7). □

3.4 Maximum Likelihood Estimation of Severity parameters

Assume that there are $z_{i,j}$ number of claims with zero payment paid in accident year i and development year j. Furthermore, knowing $r_{i,j}$ number of closed claims in accident year i and development year j, we denote the observed positive claim payments as $\{x_{i,j}^{(l)} : l = 1, 2, \ldots, r_{i,j} - z_{i,j}\}$. As we assume that $X_{i,j}^{(l)}$'s are independent and identically distributed for $l \geq 1$, the likelihood function of claim sizes given the count values is given by

$$\begin{aligned} L_{\Delta_I|D_I}(\boldsymbol{q}, \alpha, \tau) &= \prod_{i=0}^{I-1}\prod_{j=1}^{I-i} L_{\mathbf{X}_{i,j}|R_{i,1}}(q_j, \alpha, \tau) \\ &= \prod_{i=0}^{I-1}\prod_{j=1}^{I-i} q_j^{z_{i,j}} \cdot \prod_{l=1}^{r_{i,j}-z_{i,j}} \left[(1-q_j)\frac{(\frac{\alpha}{\tau\delta^{i+j-1}})^{\alpha}}{\Gamma(\alpha)}(x_{i,j}^{(l)})^{\alpha-1}e^{-\frac{\alpha}{\tau\delta^{i+j-1}}x_{i,j}^{(l)}}\right] \\ &= \prod_{i=0}^{I-1}\prod_{j=1}^{I-i} q_j^{z_{i,j}}(1-q_j)^{r_{i,j}-z_{i,j}}\left(\frac{(\frac{\alpha}{\tau\delta^{i+j-1}})^{\alpha}}{\Gamma(\alpha)}\right)^{r_{i,j}-z_{i,j}}(x_{i,j}^*)^{\alpha-1}e^{-\frac{\alpha}{\tau\delta^{i+j-1}}x_{i,j}}, \end{aligned}$$

where

$$x_{i,j} = \sum_{l=1}^{r_{i,j}-z_{i,j}} x_{i,j}^{(l)} \qquad x_{i,j}^* = \prod_{l=1}^{r_{i,j}-z_{i,j}} x_{i,j}^{(l)}.$$

Its log-likelihood function is

$$\begin{aligned} &l_{\Delta_I|D_I}(\boldsymbol{q}, \alpha, \tau,) \\ &= \sum_{i=0}^{I-1}\sum_{j=1}^{I-i}\log\left(q_j^{z_{i,j}}(1-q_j)^{r_{i,j}-z_{i,j}}\left(\frac{(\frac{\alpha}{\tau\delta^{i+j-1}})^{\alpha}}{\Gamma(\alpha)}\right)^{r_{i,j}-z_{i,j}}(x_{i,j}^*)^{\alpha-1}e^{-\frac{\alpha}{\tau\delta^{i+j-1}}x_{i,j}}\right) \\ &= \sum_{i=0}^{I-1}\sum_{j=1}^{I-i}\Big[z_{i,j}\log(q_j) + (r_{i,j}-z_{i,j})\log(1-q_j) \\ &\quad + (r_{i,j}-z_{i,j})\left[\alpha\left(\log(\alpha) - \log(\tau) - (i+j-1)\log(\delta)\right) - \log\Gamma(\alpha)\right] \\ &\quad + (\alpha-1)\log(x_{i,j}^*) - \frac{\alpha}{\tau\delta^{i+j-1}}x_{i,j}\Big]. \end{aligned} \tag{3.19}$$

The same technique is applied to get the parameter estimations as we have done in the previous section. The results are stated in the theorem below.

Theorem 3.4.1. The maximum likelihood estimation of parameter $\hat{\alpha}$ and $\hat{\delta}$ can be obtained from the following equations:

$$\log(\hat{\alpha}) - \psi(\hat{\alpha}) + 1 = \frac{\sum_{i=0}^{I-1}\sum_{j=1}^{I-i}\left[(r_{i,j}-z_{i,j})\left(\log(\hat{\tau}) + (i+j-1)\log(\delta)\right) - \log(x_{i,j}^*) + \frac{x_{i,j}}{\hat{\tau}\delta^{i+j-1}}\right]}{r_{\bullet,\bullet}^*}, \tag{3.20}$$

where $\psi(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha)$ is the digamma function, and the maximum likelihood estimations of parameters $\hat{q}_j$'s and $\hat{\tau}$ are given by

$$\hat{q}_j = \frac{\sum\limits_{i=0}^{I-j} z_{i,j}}{\sum\limits_{i=0}^{I-j} r_{i,j}}, \qquad j = 1, 2, ..., I, \tag{3.21}$$

$$\hat{\tau} = \frac{1}{r^*_{\bullet,\bullet}} \sum_{i=0}^{I-1} \sum_{j=1}^{I-i} \frac{x_{i,j}}{\delta^{i+j-1}}, \tag{3.22}$$

where

$$r^*_{\bullet,\bullet} = \sum_{i=0}^{I-1} \sum_{j=1}^{I-i} (r_{i,j} - z_{i,j}),$$

is the total observed number of closed claims with non-zero payments in the upper right triangle.

Proof. By taking the partial derivative of log-likelihood function (3.19) w.r.t. $q_{i,j}$, α and $\tau_{i,j}$, respectively, we have

$$\begin{aligned}
\frac{\partial l_{\Delta_I|D_I}}{\partial \alpha} &= (\log(\alpha) - \psi(\alpha) + 1) \cdot \sum_{i=0}^{I} \sum_{j=1}^{I-i} (r_{i,j} - z_{i,j}) \\
&\quad - \sum_{i=0}^{I} \sum_{j=1}^{I-i} \left[(r_{i,j} - z_{i,j}) \cdot (\log(\tau) + (i+j-1)\log(\delta)) - \log(x^*_{i,j}) + \frac{x_{i,j}}{\tau \delta^{i+j-1}} \right], \\
\frac{\partial l_{\Delta_I|D_I}}{\partial q_j} &= \sum_{i=0}^{I-j} \left(\frac{z_{i,j}}{q_j} - \frac{r_{i,j} - z_{i,j}}{1 - q_j} \right), \\
\frac{\partial l_{\Delta_I|D_I}}{\partial \tau} &= \sum_{j=1}^{I-i} (r_{i,j} - z_{i,j}) \left(-\frac{\alpha}{\tau} \right) + \frac{\alpha}{\tau^2 \delta^{i+j-1}} \cdot x_{i,j}.
\end{aligned}$$

Letting above three equations be zero yields (3.20)-(3.22). □

Chapter 4

Estimating Process and Prediction

In the previous chapter, the system of equations for estimating model parameters using the MLE technique has been presented. Section 4.1 presents the algorithms to estimate model parameters, and Section 4.2 presents how the estimations of model parameters can be used to predict incremental aggregate claims in cells on the lower right triangle.

4.1 Parameter Estimating Process

The maximum likelihood estimations for τ and $\boldsymbol{q}$ can be estimated explicitly from (3.21) and (3.22), respectively, with the observed data. Then, α is estimated by solving (3.20) with $\hat{\tau}$ and $\hat{\boldsymbol{q}}$. However, ρ, $\boldsymbol{\mu}$, and $\boldsymbol{\gamma}$ are estimated using an iterative process with the system of estimating equations in Theorem 3.3.1 because they are dependent to each other.

Algorithm 4.1.1. We denote $\hat{\rho}^n$, $\hat{\boldsymbol{\mu}}^n = \{\hat{\mu}_0^n, \hat{\mu}_1^n, ..., \hat{\mu}_{I-1}^n\}$, and $\hat{\boldsymbol{\gamma}}^n = \{\hat{\gamma}_0^n, \hat{\gamma}_1^n, ..., \hat{\gamma}_{I-1}^n\}$ as a set of estimated parameters in n^{th} iteration, and the algorithm of iterative process for estimating ρ, $\boldsymbol{\mu}$, and $\boldsymbol{\gamma}$ is as follows:

- First, find a set of reasonable starting values for each parameters, $\hat{\rho}^0$, $\hat{\boldsymbol{\mu}}^0$, $\hat{\boldsymbol{\gamma}}^0$.
- With the set of starting values, update the parameters following steps described below:
  1. Estimate $\hat{\rho}^1$ from (3.6) with $\hat{\rho}^0$, $\hat{\boldsymbol{\mu}}^0$, $\hat{\boldsymbol{\gamma}}^0$ as

$$\hat{\rho}^1 = \frac{\sum_{i=0}^{I-1}\left(\hat{\mu}_i^0 \sum_{j=1}^{I-i}\hat{\gamma}_{j-1}^0 + 2\sum_{j=2}^{I-i}\sum_{y=0}^{M_{i,j}}\sum_{i^*=r_{i,j}-y}^{\infty} H_{i,j}(y, i^*; \hat{\rho}^0, \hat{\mu}_i^0, \hat{\gamma}_{j-1}^0)\cdot y\right) - r_{\bullet,\bullet}}{\sum_{i=0}^{I-1}\sum_{j=1}^{I-i}\left(\hat{\mu}_i^0\hat{\gamma}_{j-1}^0 + r_{i,j-1}\right)};$$

2. Estimate $\hat{\mu}_0^1$ from (3.7) with $i=0$, $\hat{\rho}^1$, $\hat{\gamma}_0^0$, $\hat{\gamma}_1^0$, ..., $\hat{\gamma}_{I-1}^0$ and $\hat{\gamma}_I^0$ by the solving the equation,

$$\hat{\mu}_0^1 = \frac{\sum\limits_{j=1}^{I}\sum\limits_{y=0}^{M_{0,j}}\sum\limits_{i^*=r_{0,j}-y}^{\infty} H_{0,j}(y,i^*;\hat{\rho}^1,\hat{\mu}_0^1,\hat{\gamma}_{j-1}^0)\cdot i^*}{1-\hat{\gamma}_I^0}$$

where $\hat{\gamma}_I^0 = 1-\sum\limits_{j=1}^{I}\hat{\gamma}_{j-1}^0$;

3. Estimate $\hat{\gamma}_{I-1}^1$ from (3.8) with $j=I$, $\hat{\rho}^1$ and $\hat{\mu}_0^1$ by the solving the equation,

$$\hat{\gamma}_{I-1}^1 = \frac{\sum\limits_{y=0}^{M_{0,I}}\sum\limits_{i^*=r_{0,I}-y}^{\infty} H_{0,I}(y,i^*;\hat{\rho}^1,\hat{\mu}_0^1,\hat{\gamma}_{I-1}^1)\cdot i^*}{\hat{\mu}_0^1};$$

4. Estimate $\hat{\mu}_1^1$ from (3.7) with $i=1$, $\hat{\rho}^1$, γ_0^0, ..., $\hat{\gamma}_{I-1}^1$ and γ_I^0 by the solving the equation,

$$\hat{\mu}_1^1 = \frac{\sum\limits_{j=1}^{I-1}\sum\limits_{y=0}^{M_{1,j}}\sum\limits_{i^*=r_{1,j}-y}^{\infty} H_{1,j}(y,i^*;\hat{\rho}^1,\hat{\mu}_1^1,\hat{\gamma}_{j-1}^0)\cdot i^*}{1-\hat{\gamma}_{I-1}^1-\hat{\gamma}_I^0};$$

5. Estimate $\hat{\gamma}_{I-2}^1$ from (3.8) with $j=I-1$, $\hat{\rho}^1$, $\hat{\mu}_0^1$ and $\hat{\mu}_1^1$ by the solving the equation,

$$\hat{\gamma}_{I-2}^1 = \frac{\sum\limits_{i=0}^{1}\sum\limits_{y=0}^{M_{i,I-1}}\sum\limits_{i^*=r_{i,I-1}-y}^{\infty} H_{i,I-1}(y,i^*;\rho^1,\hat{\mu}_i^1,\hat{\gamma}_{I-2}^1)\cdot i^*}{\sum\limits_{i=0}^{1}\hat{\mu}_i^1};$$

6. Estimate pairs of $\{\hat{\mu}_2^1,\hat{\gamma}_{I-3}^1\}$, ..., $\{\hat{\mu}_{I-2}^1,\hat{\gamma}_1^1\}$ by repeating similar steps 4 and 5.

7. Estimate $\hat{\mu}_{I-1}^1$ from (3.7) with $i=I-1$, $\hat{\rho}^1$, $\hat{\mu}_0^1$, ..., $\hat{\mu}_{I-2}^1$, $\hat{\gamma}_0^0$, $\hat{\gamma}_1^1$, ...,$\hat{\gamma}_{I-1}^1$ by the solving the equation,

$$\hat{\mu}_{I-1}^1 = \frac{\sum\limits_{i^*=r_{I-1,1}}^{\infty} H_{I-1,1}(0,i^*;\hat{\rho}^1,\hat{\mu}_{I-1}^1,\hat{\gamma}_0^0)\cdot i^*}{1-\sum_{j=2}^{I}\hat{\gamma}_{j-1}^1-\hat{\gamma}_I^0}$$

8. Estimate $\hat{\gamma}_0^1$ from (3.8) with $j=1$, $\hat{\rho}^1$, $\hat{\mu}_0^1$, $\hat{\mu}_1^1$, ..., $\hat{\mu}_{I-1}^1$ by the solving the equation,

$$\hat{\gamma}_0^1 = \frac{\sum\limits_{i=0}^{I-1}\sum\limits_{i^*=r_{i,1}}^{\infty} H_{i,1}(0,i^*;\hat{\rho}^1,\hat{\mu}_i^1,\hat{\gamma}_0^1)\cdot i^*}{\sum_{i=0}^{I-1}\hat{\mu}_i^1}$$

9. Estimate $\hat{\gamma}_I^1$ by

$$\hat{\gamma}_I^1 = 1 - \sum_{j=1}^{I} \hat{\gamma}_{j-1}^1.$$

10. Repeat steps 1-9 by using $\hat{\rho}^1$, $\hat{\boldsymbol{\mu}}^1$, $\hat{\boldsymbol{\gamma}}^1$ as the new set of initial values to update parameter estimations until convergence.

For the initial values of the set of parameters, we first randomly choose $\hat{\rho}^0$ between 0 and 1. We denote $\widehat{W}_{i,j} = N_{i,j} - \hat{\rho}^0 \cdot N_{i,j-1}$ and estimate $\hat{\mu}_i^0$'s and $\hat{\gamma}_j^0$'s by assuming that $\widehat{W}_{i,j}$ follows the Poisson distribution for $0 \leq i \leq I, 0 \leq j \leq I-i$. Thus, equations of $\hat{\mu}_i^0$'s and $\hat{\gamma}_j^0$'s are given by

$$\begin{aligned}
\hat{\mu}_0^0 &= \sum_{j=0}^{I} \widehat{W}_{0,j}, \\
\hat{\mu}_i^0 &= \frac{\sum_{j=0}^{I-i} \widehat{W}_{i,j}}{1 - \sum_{j=I-i+1}^{I} \hat{\gamma}_j^0}, \qquad i = 1, 2, ..., I, \\
\hat{\gamma}_j^0 &= \frac{\sum_{j=0}^{I-j} \widehat{W}_{i,j}}{\sum_{i=0}^{I-j} \hat{\mu}_i^0}, \qquad j = 0, 1, ..., I.
\end{aligned}$$

4.2 Prediction

The estimation of incremental aggregate claims can be obtained based on conditional expectation. From (2.19) and (3.2), the expected incremental aggregate claims conditioning on $R_{i,j-h}$ is known and can be obtained as

$$\begin{aligned}
\mathrm{E}\left[S_{i,j} \,\middle|\, R_{i,j-h} = r_{i,j-h}\right] &= \mathrm{E}\left[\sum_{l=1}^{R_{i,j}} X_{i,j}^{(l)} \,\middle|\, R_{i,j-h} = r_{i,j-h}\right] \\
&= \mathrm{E}\left[\mathrm{E}\left[\sum_{l=1}^{R_{i,j}} X_{i,j}^{(l)} \,\middle|\, R_{i,j}\right] \,\middle|\, R_{i,j-h} = r_{i,j-h}\right] \\
&= (1-q_j)\tau \cdot \delta^{i+j-1} \cdot \mathrm{E}\left[R_{i,j} \,\middle|\, R_{i,j-h} = r_{i,j-h}\right] \\
&= (1-q_j)\tau \cdot \delta^{i+j-1} \cdot \left[\left((1-\rho)\sum_{k=1}^{h} \rho^{k-1}\gamma_{j-k}\right)\mu_i + \rho^h \cdot r_{i,j-h}\right].
\end{aligned} \tag{4.1}$$

By replacing parameters in (4.1) with estimates obtained from Chapter 3 and h with $i+j-I$, which is the most recent calender year, we can get the estimates of predicted incremental aggregate claims as

$$\begin{aligned}\hat{S}_{i,j}^{pred} &= \hat{\mathrm{E}}\left[S_{i,j} \,|\, R_{i,I-i} = r_{i,j-h}\right] \\ &= (1-\hat{q}_j)\hat{\tau} \cdot \delta^{i+j-1} \cdot \left[\left((1-\hat{\rho})\sum_{k=1}^{i+j-I} \hat{\rho}^{k-1}\hat{\gamma}_{j-k}\right)\hat{\mu}_i + \hat{\rho}^{i+j-I} \cdot r_{i,I-i}\right], \qquad (4.2) \\ &\qquad 0 \le i \le I-1,\ 1 \le j \le I,\ I+1 \le i+j \le 2(I-1).\end{aligned}$$

The total outstanding payments $TP_{i,j}$ at the end of the development year j for any particular accident year i is defined as the summation of all the payments of closed claims made in the future development year $j+1, ..., I$, and it is given by

$$TP_{i,j} = \sum_{k=j+1}^{I} S_{i,k}, \qquad j = 0, 1, ...I.$$

Thus, the total outstanding payments at the current calender year for the accident year i, which is development year $I - I$ can be estimated by

$$\widehat{TP}_{i,I-i}^{pred} = \sum_{k=I-i+1}^{I} \hat{S}_{i,k}^{pred}, \qquad 1 \le i \le I. \qquad (4.3)$$

Note that $\widehat{TP}_{0,I}^{pred} = 0$.

4.3 Means Square Error of Prediction

We employ the mean square error of prediction (MSEP) to measure the accuracy of the prediction for incremental aggregate claims $S_{i,j}$. MSEP is commonly used to assess the accuracy of predictions in the claims reserving modeling; see for example, Wüthrich (2003), Meng et al. (2018), Bai (2016). We define the mean square error of prediction of our prediction as

$$\begin{aligned}\mathrm{MSEP}\left[\hat{S}_{i,j}^{pred} \,|\, D_I\right] &= \mathrm{E}\left[\left(S_{i,j} - \hat{S}_{i,j}^{pred}\right)^2 \,\Big|\, D_I\right] \\ &= \mathrm{Var}\left[S_{i,j} \,|\, D_I\right] + \mathrm{E}\left[\left(\hat{S}_{i,j}^{pred} - \mathrm{E}\left[S_{i,j} \,|\, D_I\right]\right)^2 \Big| D_I\right], \qquad (4.4) \\ &\qquad 0 \le i \le I-1,\ 1 \le j \le I,\ I+1 \le i+j \le 2(I-1).\end{aligned}$$

The mean square error of the prediction is decomposed to two terms as shown in (4.4). The first term is the prediction error, and second term is the estimation error. As the estimation error is difficult to be expressed clearly in a closed form and estimated, we consider the prediction error term only in this project to assess the level of accuracy of prediction. Thus, we can approximate MSEP as

$$\mathrm{MSEP}\left[\hat{S}_{i,j}^{pred} \mid D_I\right] \approx \mathrm{Var}\left[S_{i,j} \mid D_I\right]. \tag{4.5}$$

In the case that if the model parameters are known, we can estimate the MSEP using (4.5) as follows:

$$\begin{aligned}
&\mathrm{Var}\left[S_{i,j} \mid D_I\right] \\
&\quad= \left(\tau\delta^{i+j-1}\left(1-q_j\right)\right)^2 \cdot \left[(1-\rho)\left(\sum_{k=1}^{i+j-I} \rho^{k-1} \cdot \gamma_{j-k}\right)\mu_i + \rho^{i+j-I}(1-\rho^{i+j-I}) \cdot R_{i,I-i}\right] \\
&\qquad+ \left[(1-\rho)\left(\sum_{k=1}^{i+j-I} \rho^{k-1} \cdot \gamma_{j-k}\right)\mu_i + \rho^{i+j-I} \cdot R_{i,I-i}\right] \cdot (1-q_j) \cdot \left(\tau\delta^{i+j-1}\right)^2 \left(\frac{1}{\alpha} + q_j\right),
\end{aligned}$$

If the model parameters are unknown, then (4.5) can be estimated by

$$\begin{aligned}
&\widehat{\mathrm{Var}}\left[S_{i,j} \mid D_I\right] \\
&\quad= \left(\hat{\tau}\delta^{i+j-1}\left(1-\hat{q}_j\right)\right)^2 \cdot \left[(1-\hat{\rho})\left(\sum_{k=1}^{i+j-I} \hat{\rho}^{k-1} \cdot \hat{\gamma}_{j-k}\right)\hat{\mu}_i + \hat{\rho}^{i+j-I}(1-\hat{\rho}^{i+j-I}) \cdot R_{i,I-i}\right] \\
&\qquad+ \left[(1-\hat{\rho})\left(\sum_{k=1}^{i+j-I} \hat{\rho}^{k-1} \cdot \hat{\gamma}_{j-k}\right)\hat{\mu}_i + \hat{\rho}^{i+j-I} \cdot R_{i,I-i}\right] \cdot (1-\hat{q}_j) \cdot \left(\hat{\tau}\delta^{i+j-1}\right)^2 \left(\frac{1}{\hat{\alpha}} + \hat{q}_j\right).
\end{aligned} \tag{4.6}$$

For the total outstanding payments, MSEP is also used to measure the accuracy of predictions. According to (4.3), we define the MSEP of $\widehat{TP}_{i,I-i}^{pred}$ for $1 \le i \le I$ as

$$\begin{aligned}
\mathrm{MSEP}\left[\widehat{TP}_{i,I-i}^{pred}\right] &= \mathrm{E}\left[\left(TP_{i,I-i} - \widehat{TP}_{i,I-i}^{pred}\right)^2\right] \\
&= \mathrm{Var}\left[\sum_{k=I-i+1}^{I} S_{i,k}\right] + \mathrm{E}\left[\left(\sum_{k=I-i+1}^{I} \hat{S}_{i,k}^{pred} - \sum_{k=I-i+1}^{I} \mathrm{E}\left[S_{i,k}\right]\right)^2\right].
\end{aligned} \tag{4.7}$$

The first and second terms in (4.7) are prediction and estimation errors, respectively. As the estimation error is difficult to be estimated, we consider the prediction error term only, and thus we can approximate the MSEP as

$$\mathrm{MSEP}\left[\widehat{TP}_{i,I-i}^{pred}\right] \approx \mathrm{Var}\left[\sum_{k=I-i+1}^{I} S_{i,k}\right]. \tag{4.8}$$

If the model parameters are known, then MSEP can be estimated using (4.8) as

$$\text{MSEP}\left[\widehat{TP}^{pred}_{i,I-i}\right] = \sum_{k=I-i+1}^{I} (1-q_k)\cdot\left(1+\frac{1}{\alpha}\right)\cdot\left(\sum_{n=1}^{k}(1-\rho)\cdot\rho^{n-1}\cdot\gamma_{k-n}\right)\cdot\mu_i\cdot\left(\tau\delta^{i+k-1}\right)^2.$$

If the model parameters are unknown, we estimate the MSEP by replacing the parameters with estimations of the parameters, and it is given by

$$\widehat{\text{MSEP}}\left[\widehat{TP}^{pred}_{i,I-i}\right] = \sum_{k=I-i+1}^{I} (1-\hat{q}_k)\cdot\left(1+\frac{1}{\hat{\alpha}}\right)\cdot\left(\sum_{n=1}^{k}(1-\hat{\rho})\cdot\hat{\rho}^{n-1}\cdot\hat{\gamma}_{k-n}\right)\cdot\hat{\mu}_i\cdot\left(\hat{\tau}\delta^{i+k-1}\right)^2.$$

Chapter 5

Numerical Illustrations

In this chapter, we conduct a simulation study to examine the accuracy of the model parameter estimations and the error of the predictions. We simulate data, the loss development triangles for counts and sizes based on the counts, with two different sizes (i.e., different values of I) to understand the impact of the triangle size on the model performance. Section 5.1 presents the estimations of model parameters under the simulation study, and Section 5.2 illustrates the prediction of aggregate incremental payments and shows the prediction errors using a simulated sample data.

5.1 Estimation of Model Parameters

The simulation study is conducted for two different sizes of triangle $I = 6$ and $I = 10$ for this project. To simulate data, we first assume that the inflation rate is constant at 3%, and choose $\rho = 0.5$ and $\mu_i = 200$ for all $i = 0, 1, ..., I$. As the number of claims are non-increasing in general, we choose the values of γ_j's in a non-increasing pattern and that satisfy the assumption $\sum_{j=0}^{I} \gamma_j = 1$. The values of parameters used for the simulation study are provided in Table 5.1. We generate 1000 sets of data in total for each of different sizes to evaluate the accuracy of parameter estimations.

True Parameters		
ρ	0.5	
μ_i	200	for $0 \leq i \leq I$
γ_j	$I = 6$;	0.5, 0.2, 0.1, 0.1, 0.05, 0.03, 0.02
	$I = 10$;	0.3, 0.2, 0.1, 0.1, 0.1, 0.08, 0.07, 0.03, 0.01, 0.005, 0.005
q_j	0.01	for $0 \leq j \leq I$
α	2.0	
τ	1000	

Table 5.1: True parameter used to generate sets of samples for $R_{i,j}$ and $X_{i,j}^{(l)}$ for $i + j \leq I$

In this project, we choose the closed rate ρ for 0.5 i.e. the 50% of reported claims from the previous development year are closed. The average size of individual claims in cell $(0,1)$ is 200, and by (2.22), average size of individual claims in cell (i,j) is $200\delta^{i+j-1}$ for $0 \le i \le I$, $0 \le j \le I$ and $i+j \le I$. For the closed claim counts, 99% of claims have non-zero payments.

5.1.1 Relative Bias and Mean Square Error

In order to compare estimations of different model parameters, we employ relative bias which is defined as

$$\text{Rel. Bias}(\hat{\theta}) = \sum_{n=1}^{N} \frac{\left(\hat{\theta}_n - \theta\right)}{N \cdot \theta}.$$

where $\hat{\theta}_n$ is the estimation of the true parameter θ obtained from the n^{th} simulated data, and N is the total number of data sets simulated. We also use the root of mean square error (RMSE) to measure the accuracy of estimations as it uses the same units as the model parameters. The RMSE is given by

$$\text{RMSE}(\hat{\theta}) = \sqrt{\sum_{n=1}^{N} \frac{\left(\hat{\theta}_n - \theta\right)^2}{N}}$$

The relative bias and RMSE of the estimations are shown in Tables 5.2 and 5.3 for $I = 6$ and $I = 10$. As we have stated Assumption 2.3.1 that the newly reported claims are not closed within the same year, there are no closed claims at the beginning of the development periods so that $r_{i,0} = 0$ for $0 \le i \le I$. Thus, we do not have estimation results for μ_I in this section. As γ_I is not being used for any prediction, we do not contain the estimations for γ_I; however, it can be estimated from Assumption 2.3.1. Note that there is no observation available to estimate μ_I; however, we can apply the regression using $\hat{\mu}_0, ..., \hat{\mu}_{I-1}$ to get $\hat{\mu}_I$

From Tables 5.2 and 5.3, we first observe that there is an improvement on the estimations of ρ, in the sense that both the relative bias and RMSE are decreasing from $I = 6$ to $I = 10$. Moreover, estimations of γ_j's and μ_i's are relatively better for $I = 10$ than that for $I = 6$ in terms of RMSE. This improvement may come from the fact that a larger size (dimension) of the triangle contains more information (observations) generally. For the estimations of μ_i's, we have the greatest RMSE for the most recent accident year $I - 1$ for both cases because of less information being used for estimating μ_{I-1}. Similarly, we observe that the RMSE of estimation of μ_i's for $I = 10$ is decreasing and then increasing. Therefore, the RMSE of $\hat{\mu}_i$'s are not guaranteed to get greater for the recent accident years.

For the MLEs for parameters used in modeling of payment size, the estimations of α and τ for $I = 10$ have better accuracy measured by the relative bias and RMSE because that more observations are available for a larger I.

	$I = 6$				
	Closed Claim Count			Size of Payment	
	Rel. Bias	RMSE		Rel. Bias	RMSE
$\hat{\rho}$	-0.16951	0.11495	$\hat{\alpha}$	0.00703	0.09432
$\hat{\mu}_0$	0.05521	23.72327			
$\hat{\mu}_1$	0.05291	24.43121			
$\hat{\mu}_2$	0.04286	23.12727			
$\hat{\mu}_3$	0.03788	24.89677	$\hat{\tau}$	-0.00039	24.73076
$\hat{\mu}_4$	0.03220	27.17824			
$\hat{\mu}_5$	0.03957	34.27608			
$\hat{\mu}_6$	—	—			
$\hat{\gamma}_0$	-0.15857	0.10537	—	—	—
$\hat{\gamma}_1$	0.00581	0.03127	$\hat{q}_1$	-0.02514	0.00587
$\hat{\gamma}_2$	0.09574	0.02503	$\hat{q}_2$	-0.01104	0.00676
$\hat{\gamma}_3$	0.00542	0.02187	$\hat{q}_3$	0.00297	0.00876
$\hat{\gamma}_4$	-0.04093	0.00809	$\hat{q}_4$	0.03624	0.01111
$\hat{\gamma}_5$	0.23683	0.02219	$\hat{q}_5$	0.03461	0.01762
$\hat{\gamma}_6$	—	—	$\hat{q}_6$	0.12907	0.03307

Table 5.2: Relative bias and square root of mean square error of parameter estimation of the number of closed claim and the size of payments when $I = 6$.

	$I = 10$				
	Closed Claim Count			Size of Payment	
	Rel. Bias	RMSE		Rel. Bias	RMSE
$\hat{\rho}$	-0.12666	0.09678	$\hat{\alpha}$	0.00324	0.07184
$\hat{\mu}_0$	0.07230	23.56722			
$\hat{\mu}_1$	0.06817	22.55957			
$\hat{\mu}_2$	0.06931	22.83104			
$\hat{\mu}_3$	0.05739	21.53735			
$\hat{\mu}_4$	0.05913	23.51439			
$\hat{\mu}_5$	0.05003	22.86506	$\hat{\tau}$	-0.00018	18.65888
$\hat{\mu}_6$	0.04810	24.40884			
$\hat{\mu}_7$	0.05029	26.34685			
$\hat{\mu}_8$	0.03925	27.96352			
$\hat{\mu}_9$	0.04797	42.08275			
$\hat{\mu}_{10}$	—	—			
$\hat{\gamma}_0$	-0.13936	0.05993	—	—	—
$\hat{\gamma}_1$	-0.05774	0.01460	$\hat{q}_1$	-0.01197	0.00596
$\hat{\gamma}_2$	0.02604	0.01924	$\hat{q}_2$	0.02799	0.00561
$\hat{\gamma}_3$	-0.00147	0.01730	$\hat{q}_3$	-0.01347	0.00678
$\hat{\gamma}_4$	-0.03204	0.00977	$\hat{q}_4$	-0.00354	0.00770
$\hat{\gamma}_5$	-0.01605	0.00884	$\hat{q}_5$	0.00525	0.00883
$\hat{\gamma}_6$	-0.01730	0.00844	$\hat{q}_6$	0.03588	0.01055
$\hat{\gamma}_7$	-0.19362	0.00581	$\hat{q}_7$	0.02174	0.01267
$\hat{\gamma}_8$	-0.02859	0.00029	$\hat{q}_8$	0.06328	0.01803
$\hat{\gamma}_9$	0.21940	0.00110	$\hat{q}_9$	0.03200	0.02890
$\hat{\gamma}_{10}$	—	—	$\hat{q}_{10}$	-0.08051	0.05011

Table 5.3: Relative bias and square root of mean square error of parameter estimation of the number of closed claim and the size of payments when $I = 10$.

Since the estimation of q_j's are only depending on past experience of development years, the estimations of q_j have larger RMSE for a larger j because we only have few observations available for estimating q_j's. Thus, we can observe that RMSE is increasing as j gets bigger for both $I = 6$ and $I = 10$.

To conclude, the estimations of parameters are generally improved from $I = 6$ to $I = 10$ in terms of relative bias and RMSE because there are more information available for estimating parameters.

5.1.2 Distribution of $\hat{\rho}$

We present the histogram of $\hat{\rho}$ in Figure 5.1 and box plot of $\hat{\rho}$ in Figure 5.2 for $I = 6$ and $I = 10$ together to see the difference more closely. From the histogram, we first observe that the distribution of $\hat{\rho}$ for $I = 6$ is skewed to the left while the distribution of that for $I = 10$ is skewed to the right and more centered around the true value. From the box plot, we could see that despite a larger interquartile, the number of outliers is less for $I = 10$; thus, the RMSE is smaller for bigger value of I. From our study, it seems that the accuracy of the estimation is improved. However, $I > 10$ case is not practically reasonable as annually recorded data may not be consistent for a longer period.

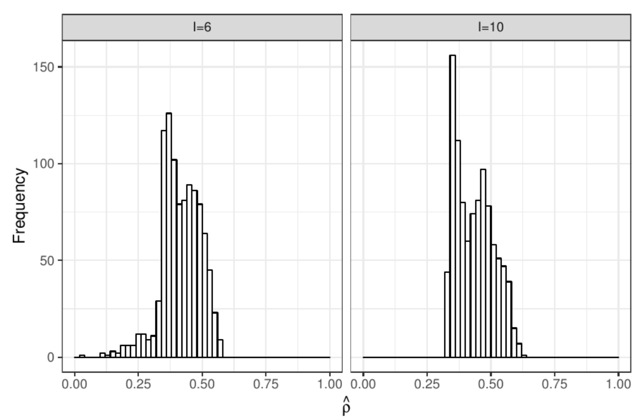

Figure 5.1: Histogram of $\hat{\rho}$.

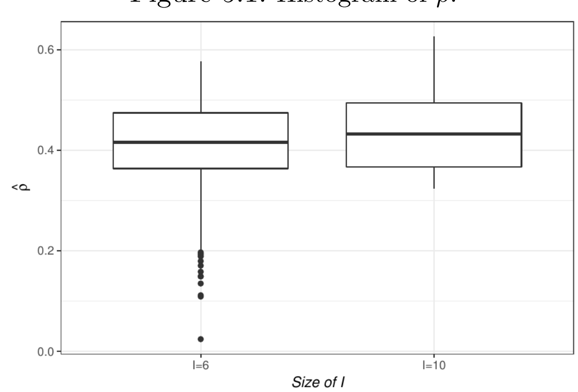

Figure 5.2: Box plot of $\hat{\rho}$.

5.1.3 Distribution of $\hat{\mu}_i$

In this section, we present separate histograms of $\hat{\mu}_i$ in Figure 5.3 and 5.5, and box plots in Figure 5.4 and 5.6 of $\hat{\mu}_i$ for $i = 0, 1, ..., I-1$, for $I = 6$ and $I = 10$, respectively. From the histograms, we can observe that $\hat{\mu}_i$'s are centered around the true value of μ_i with a small positive skewness. One observation in common for both cases is that the distribution of $\hat{\mu}_i$ is more spread out for bigger i values. As we have discussed in Section 5.1.1, this is because that less information available (observed) for recent accident years. Additionally, as $\hat{\mu}_{I-1}$ is updated at very last after updating other model parameters, it happens to have a larger variation based on estimations of other parameters. The positive skewness is also shown in the box plots as the median is above the true value of μ_i, implying over-estimation.

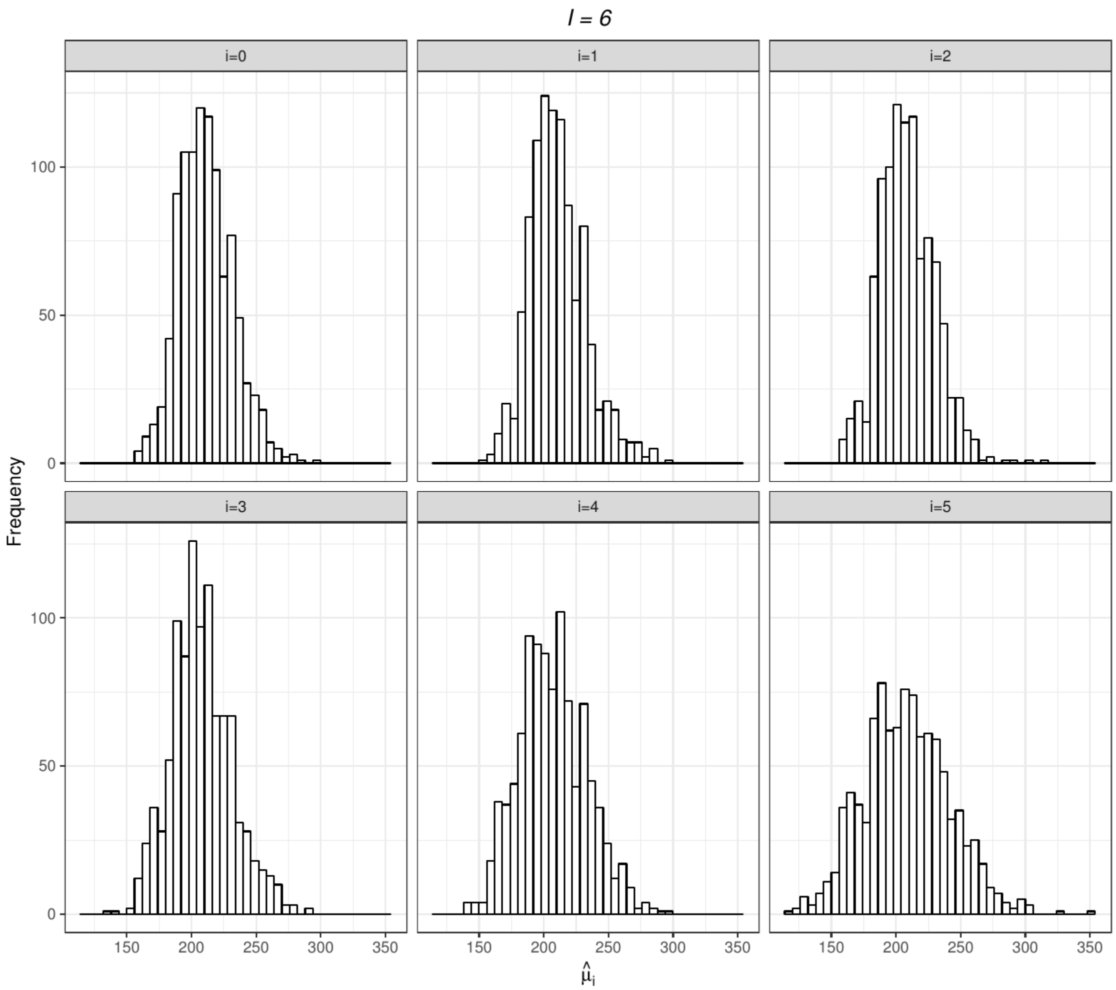

Figure 5.3: Histogram of $\hat{\mu}_i$ when $I = 6$.

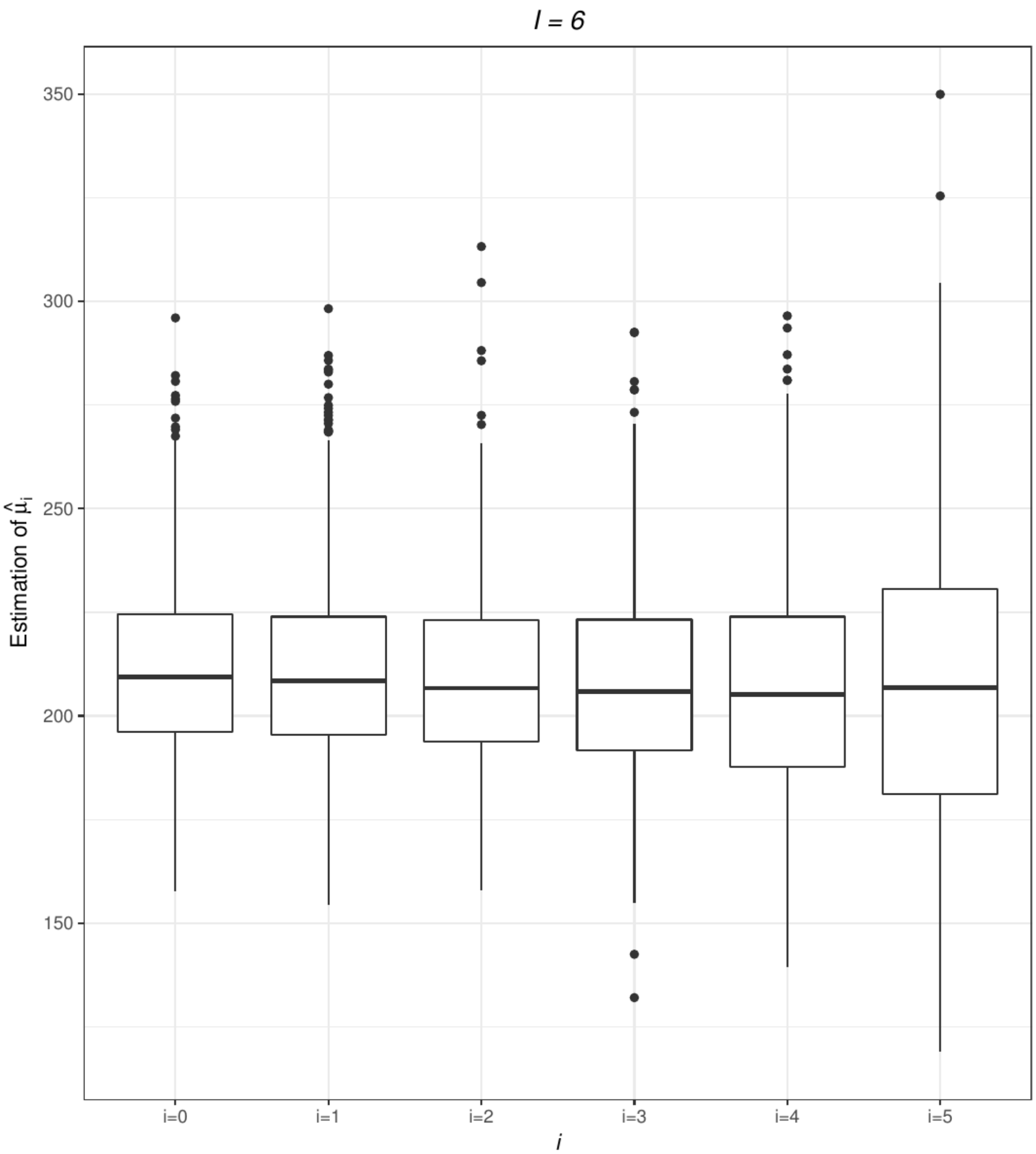

Figure 5.4: Box plot of $\hat{\mu}_i$ when $I = 6$.

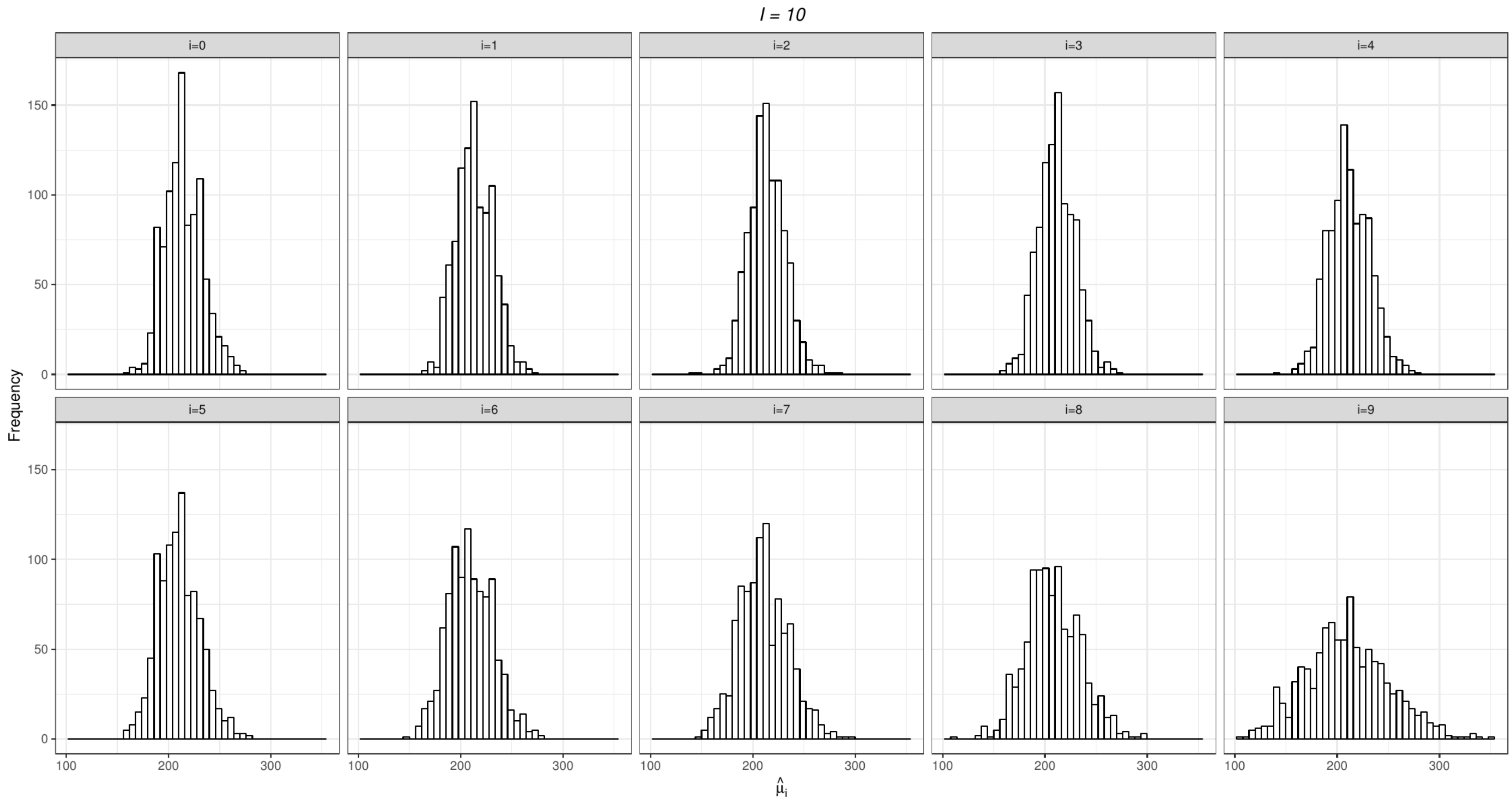

Figure 5.5: Histogram of $\hat{\mu}_i$ when $I = 10$.

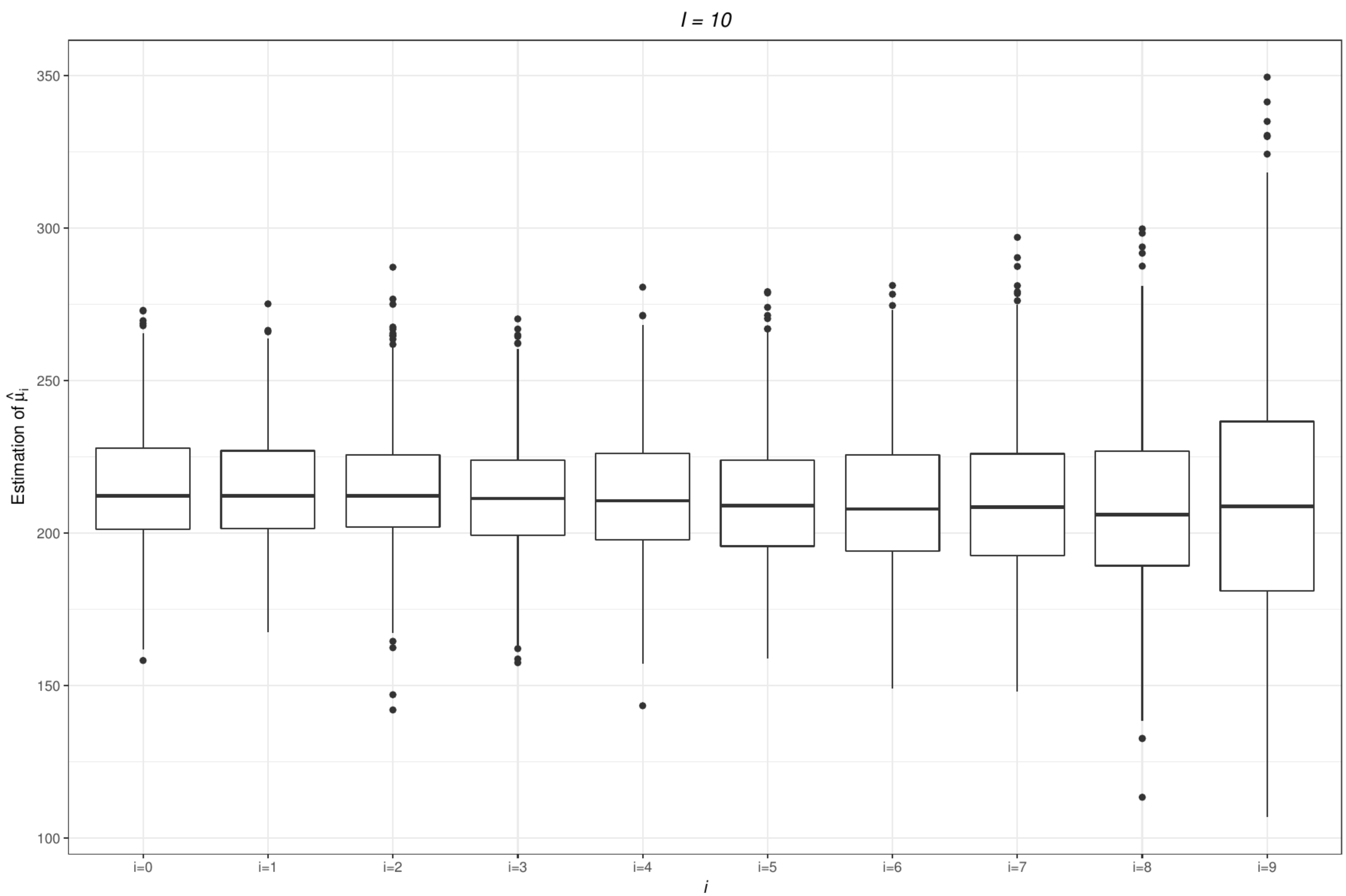

Figure 5.6: Box plot of $\hat{\mu}_i$ when $I = 10$.

5.1.4 Distribution of $\hat{\alpha}, \hat{\tau}$ and $\hat{q}_j$

The numerical results for estimations of parameters in the mixed gamma distribution are discussed in this subsection. Two different types of plot for $\hat{\alpha}$ and $\hat{\tau}$ for both $I = 6$ and $I = 10$ are shown in Figures 5.7 - 5.10. All the figures show that estimations of α and τ are centered around their true values. By increasing the size of the triangle from $I = 6$ to $I = 10$, they have higher peaks at their true values and smaller ranges of interquartile. Thus, we can see that $\hat{\alpha}$ and $\hat{\tau}$ have higher accuracy for a larger size of the triangle. The distribution of q_j are shown in Figures 5.11 - 5.14. We can observe that the frequency of values of estimations around 0 is increasing for larger values of j. In addition, the box plots show that the number of outliers is increasing significantly for larger values of j because there is less information (observations) for claims. Although the estimations of parameters of individual claims for $I = 6$ is reasonably good, we observe that there is improvement in accuracy of estimations for $I = 10$ in terms of relative bias and RMSE. One thing to notice from this study is that $\hat{q}_j$ varies more significantly depending on a development year j rather than the size of the triangle.

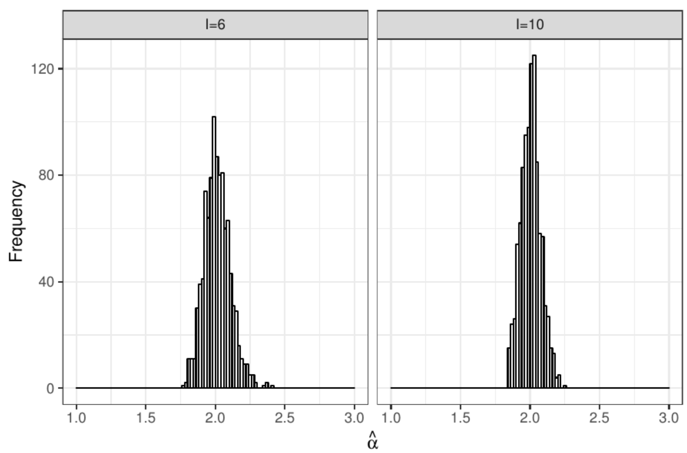

Figure 5.7: Histogram of $\hat{\alpha}$.

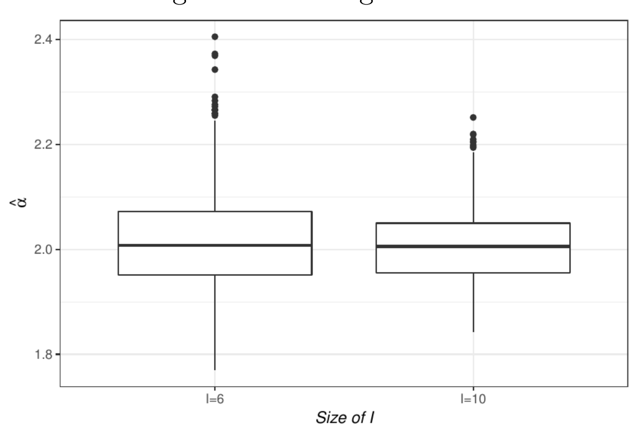

Figure 5.8: Box plot of $\hat{\alpha}$.

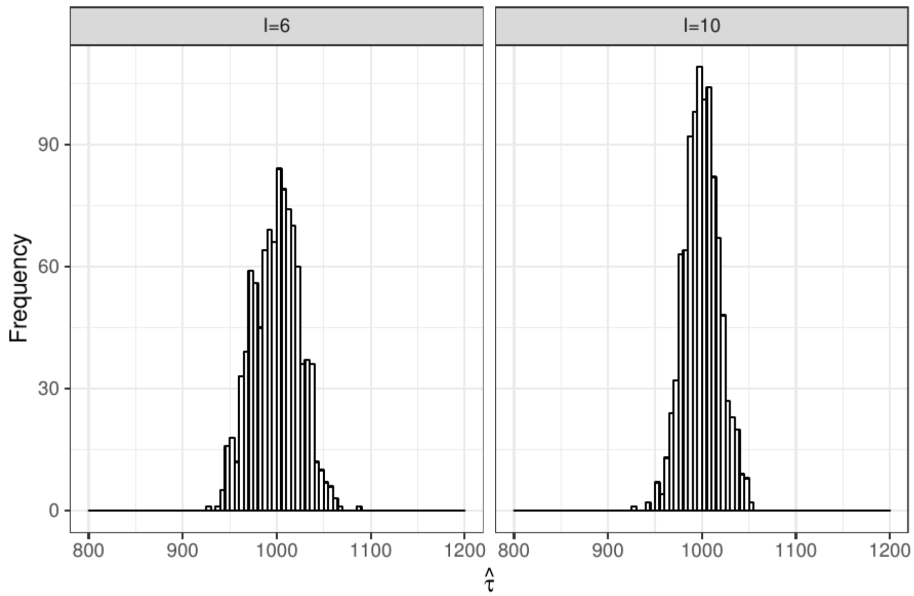

Figure 5.9: Histogram of $\hat{\tau}$.

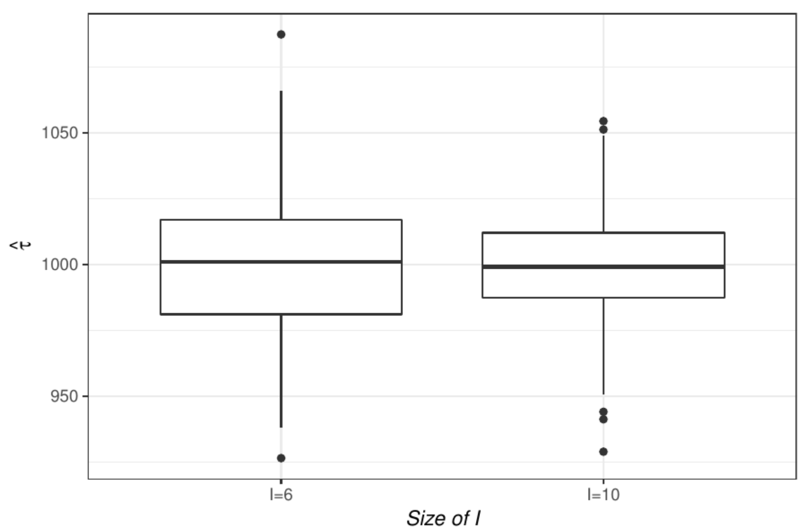

Figure 5.10: Box plot of $\hat{\tau}$.

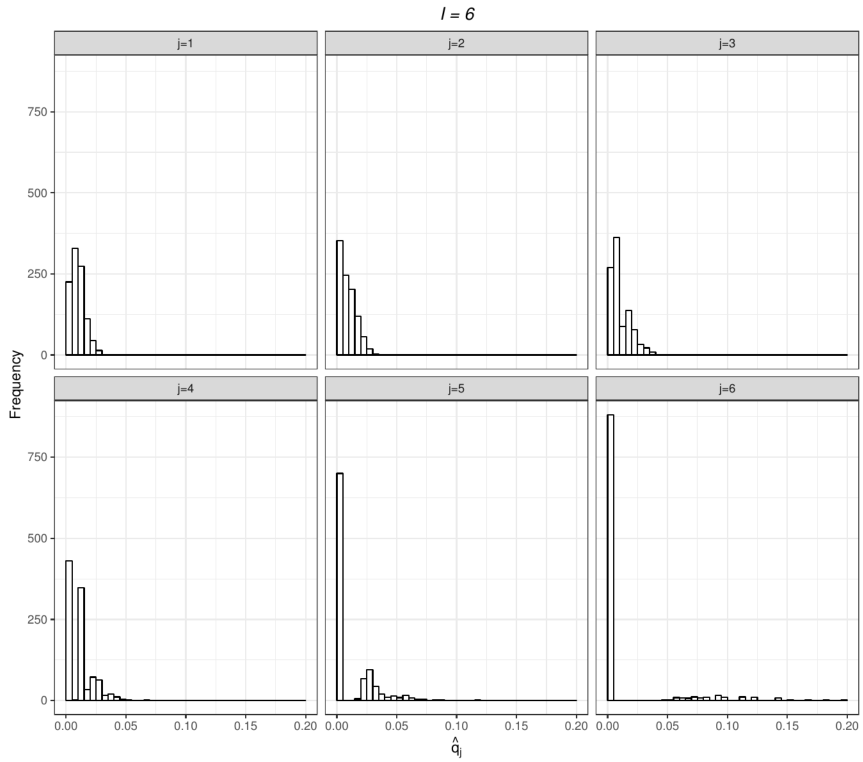

Figure 5.11: Histogram of $\hat{q}_j$ when $I = 6$.

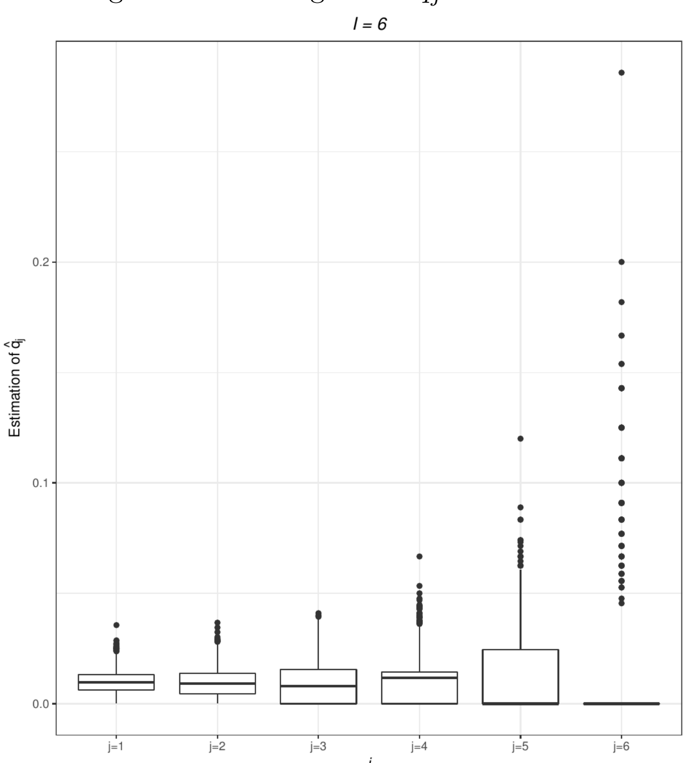

Figure 5.12: Box plot of $\hat{q}_j$ when $I = 6$.

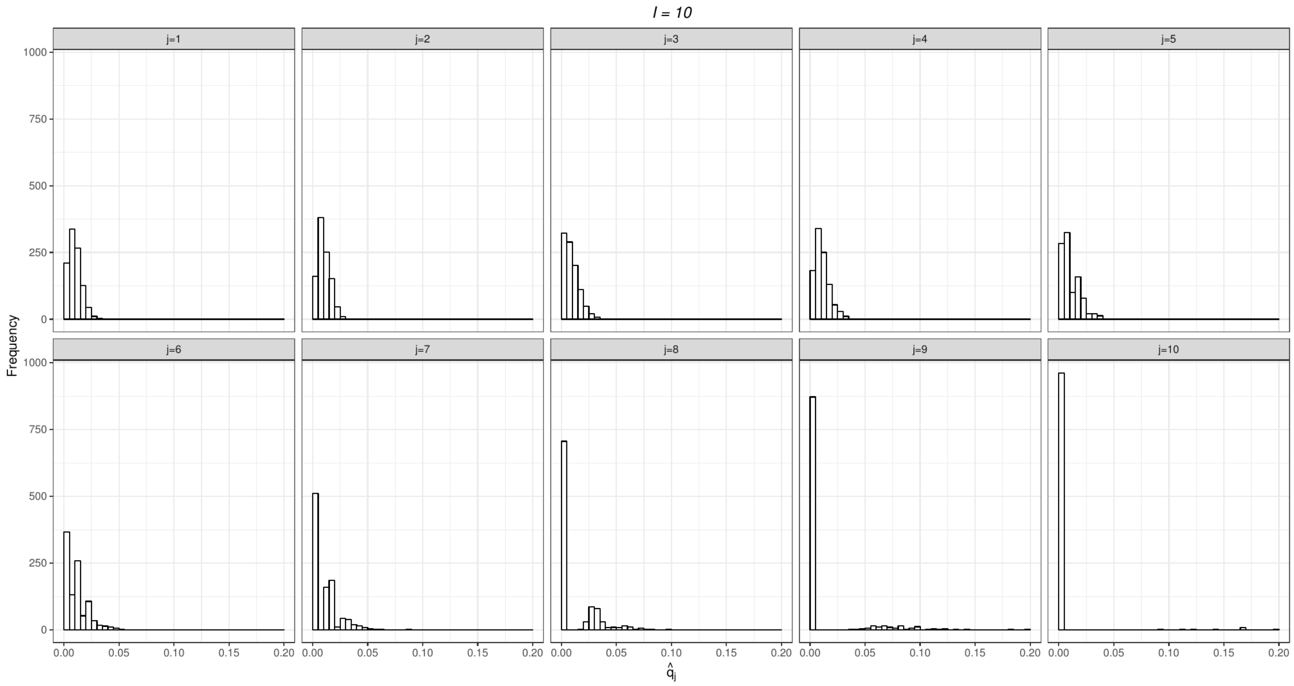

Figure 5.13: Histogram of $\hat{q}_j$ when $I = 10$.

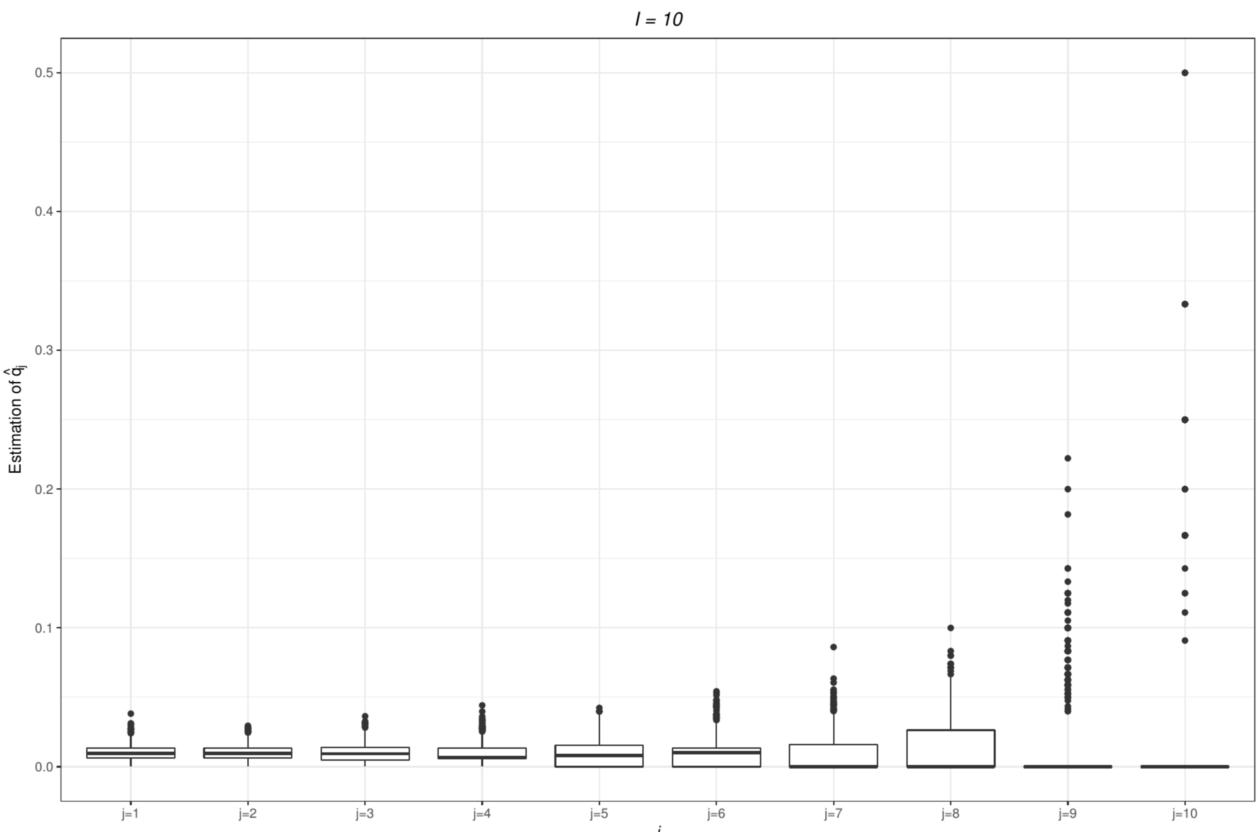

Figure 5.14: Box plot of $\hat{q}_j$ when $I = 10$.

5.2 Prediction of Aggregate Loss Reserve

First, We generate a sample of observations for the number of closed claims and payment sizes by simulation under the same set of true parameters used in the previous section shown in Table 5.1 with a constant inflation rate of 3%. We then get the prediction and MSEP of aggregate payments for each cell (i, j) such that $0 \leq i \leq I-1,\ 1 \leq j \leq I,$ and $I+1 \leq i+j \leq 2(I-1)$ based on the estimated parameters provided in Tables A.1 and A.2 using (4.2) and (4.6). Tables 5.4 and 5.6 show estimations of future aggregate incremental claims for $I = 6$ and $I = 10$, and Tables 5.5 and 5.7 show the values of square root of mean square error, respectively. In order to compare the accuracy of predictions in each cell (i, j), we also provide the value of MSEP in percentage by dividing the square root of mean square error of prediction by its estimated value. Since the total outstanding payments in current calender year describes the current financial status, we also present the estimations and prediction errors for total outstanding payments in Tables 5.8 and 5.9. Second, for $I = 10$, we generate a sample of observations using $\mu_i = 1000$ for all i while other parameters remain same to understand the impact of the μ_i values on the predictions for different number of closed claims. The results for this study are shown in Tables 5.10 - 5.12.

From Table 5.5 we can see that the prediction error (in percentage) increases as j increases for particular accident year i. This is because we have less observed data for late development years which results in more uncertainty in estimations. On the other hand, for Table 5.7, the prediction error (in percentage) increases as j increases up to $j = 6$ and then it drops at $j = 7$. Then again it increases as j increases. The reason for this could be explained by combining the estimation process and the availability of observed information. According to the estimation algorithm discussed in Section 4.1, for each iteration, $\hat{\gamma}_{I-k}$ is updated using $\hat{\rho}, \hat{\mu}_0, ...\hat{\mu}_k$, and this may increase the uncertainty by having more number of estimations to update $\hat{\gamma}_{I-k}$ as k increases. However, at the same time, there is more observed information to estimate $\hat{\gamma}_{I-k}$ for a larger k (smaller $I-k$) which can possibly result in a more accurate estimation. For these reasons, the prediction error (in percentage) of incremental aggregate claims has an increasing trend for increasing j but have a drop at $j = 7$.

From Tables 5.8 and 5.9. we can observe that there is relatively bigger uncertainty for a small i which have the small outstanding payments. This comes from the fact there is only little information observed to estimate γ_j and q_j for large j. On the other hand, as we have more information for a larger i to estimate γ_j and q_j for small j, the uncertainty coming from claims for a large i with large j has less influence on the total outstanding payments. Thus, the total outstanding payments have smaller MSEP in percentage. In conclusion, we have reasonably good results in predictions with the acceptable range of errors. However, as we do not consider the estimations error for MSEP, there can be large MSEP of total outstanding payments especially for claims for a large i because of more number of estimations.

The same phenomenon is observed in Table 5.11 and 5.12 as we discussed previously for Table 5.7 and 5.9. However, we can find one interesting result from the comparison between different μ_i values. The relative errors of prediction for $S_{i,j}$ and $TP_{i,I-i}$ are significantly reduced from $\mu_i = 200$ to $\mu_i = 1000$. The reason for the reduction in the relative error of prediction is because the sample of observations using larger values of μ_i's generate larger values of $r_{i,j}$'s which are the information (observations) used in the parameter estimations. Thus, the level of accuracy increases as $r_{i,j}$'s increase in terms of relative prediction error.

Total Incremental Payments $\hat{S}_{i,j}^{pred}$						
Accident Year i	Development year j					
	0	1	2	3	4	5
0	0	49322	45879	34372	28205	19944
1	0	51155	47389	35873	28741	20687
2	0	52831	48759	36925	29848	**20876**
3	0	54269	50322	36889	**29758**	**20928**
4	0	55729	51772	**38413**	**31007**	**21778**
5	0	57096	**53030**	**39470**	**31899**	**22418**

Table 5.4: The estimated aggregate incremental payments, $I = 6$ and $\mu_i = 200 \ \forall \ i$.

$\widehat{\mathbf{MSEP}}^{1/2}\left[\hat{S}_{i,j}^{pred} \mid D_I\right]$						
Accident Year i	Development year j					
	0	1	2	3	4	5
0						
1						
2						**4843**
3					**5623**	**4861**
4				**7102**	**6047**	**4841**
5			**7411**	**7363**	**5732**	**4455**
in %						
0						
1						
2						**0.232**
3					**0.189**	**0.232**
4				**0.185**	**0.195**	**0.222**
5			**0.140**	**0.187**	**0.180**	**0.199**

Table 5.5: The estimated mean square error of predictions, $I = 6$ and $\mu_i = 200 \ \forall \ i$.

Total Incremental Payments $\hat{S}_{i,j}^{pred}$										
Accident Year i	Development year j									
	0	1	2	3	4	5	6	7	8	9
0	0	29705	35807	28454	25857	24667	21589	20146	13505	8739
1	0	30595	36792	29276	26907	25331	22478	19508	13776	8994
2	0	31504	38071	31494	27511	25706	22987	21283	14307	**9099**
3	0	32670	39102	31051	28409	26755	23954	20623	**14073**	**9115**
4	0	33343	40156	31818	29119	27616	24715	**21818**	**14713**	**9487**
5	0	34419	41237	32802	30299	28280	**25135**	**22422**	**15139**	**9768**
6	0	35667	42536	34096	31265	**29153**	**25850**	**23089**	**15596**	**10064**
7	0	36470	44141	34962	**31774**	**29955**	**26600**	**23775**	**16062**	**10366**
8	0	37533	44795	**35996**	**32795**	**30925**	**27465**	**24548**	**16585**	**10704**
9	0	38537	**46634**	**37126**	**33834**	**31908**	**28339**	**25330**	**17113**	**11045**

Table 5.6: The estimated aggregate incremental payments, $I = 10$ and $\mu_i = 200 \; \forall \; i$.

$\widehat{\textbf{MSEP}}^{1/2}\left[\hat{S}_{i,j}^{pred} \,\middle|\, D_I\right]$										
Accident Year i	Development year j									
	0	1	2	3	4	5	6	7	8	9
0										
1										
2										**3387**
3									**4406**	**3200**
4								**5187**	**4182**	**2869**
5							**6461**	**5108**	**3983**	**2706**
6						**6858**	**6711**	**4997**	**3916**	**2661**
7					**7445**	**7180**	**6828**	**5016**	**3952**	**2688**
8				**7981**	**7958**	**7364**	**6997**	**5120**	4044	2752
9			**7795**	**8062**	**8002**	**7468**	**7155**	**5237**	**4147**	**2823**
in %										
0										
1										
2										**0.372**
3									**0.313**	**0.351**
4								**0.238**	**0.284**	**0.302**
5							**0.257**	**0.228**	**0.263**	**0.277**
6						**0.235**	**0.260**	**0.216**	**0.251**	**0.264**
7					**0.234**	**0.240**	**0.257**	**0.211**	**0.246**	**0.259**
8				**0.222**	**0.243**	**0.238**	**0.255**	**0.209**	**0.244**	**0.257**
9			**0.167**	**0.217**	**0.237**	**0.234**	**0.252**	**0.207**	**0.242**	**0.256**

Table 5.7: The estimated mean square error of predictions, I=10 and $\mu_i = 200 \; \forall \; i$.

Total Outst. Payments			
Accident Year i	$\mathbf{TP}_{i,I-i}^{pred}$	$\mathbf{MSEP}^{1/2}$	**in%**
0	—	—	—
1	13749	4891	0.356
2	34822	7885	0.226
3	64721	10915	0.169
4	105792	13994	0.132
5	161844	17415	0.108

Table 5.8: The estimated total outstanding payments and MSEP

Total Outst. Payments			
Accident Year i	$\mathbf{TP}_{i,I-i}^{pred}$	$\mathbf{MSEP}^{1/2}$	**in%**
0	—	—	—
1	5038	3139	0.623
2	14225	5354	0.376
3	28353	7660	0.270
4	51381	10310	0.201
5	77989	12779	0.164
6	109445	15229	0.139
7	144396	17609	0.122
8	185073	20058	0.108
9	237575	22840	0.096

Table 5.9: The estimated total outstanding payments and MSEP, I=10 and $\mu_i = 200 \; \forall \; i$.

Total Incremental Payments $\hat{S}_{i,j}^{pred}$										
	Development year j									
Accident Year i	0	1	2	3	4	5	6	7	8	9
0	0	148562	178397	145235	129090	121744	107625	97312	68576	41536
1	0	153382	183532	149776	132467	124870	112300	99929	70205	42405
2	0	156744	188708	152775	135215	130217	116140	104003	72842	**48046**
3	0	162500	195110	157377	140618	132271	119310	105705	**73460**	**48792**
4	0	167268	201263	161950	144983	137442	121319	**108318**	**75154**	**49918**
5	0	172681	207286	166370	149477	140557	**125048**	**111557**	**77401**	**51410**
6	0	177782	212479	172335	153192	**145238**	**128979**	**115055**	**79827**	**53021**
7	0	182303	219293	178003	**158434**	**149675**	**132939**	**118601**	**82289**	**54657**
8	0	187621	227049	**183239**	**163028**	**154171**	**137015**	**122292**	**84855**	**56361**
9	0	192889	**231225**	**186467**	**166009**	**157080**	**139648**	**124675**	**86511**	**57461**

Table 5.10: The estimated aggregate incremental payments, $I = 10$ and $\mu_i = 1000 \; \forall \; i$.

$\widehat{\mathbf{MSEP}}^{1/2}\left[\hat{S}_{i,j}^{pred} \,\middle|\, D_I\right]$										
Accident Year i	Development year j									
	0	1	2	3	4	5	6	7	8	9
0										
1										
2										**7437**
3									**9001**	**8370**
4								**9954**	**8967**	**7896**
5							**12777**	**10200**	**8510**	**7298**
6						**13452**	**14100**	**9668**	**7635**	**6452**
7					**14937**	**15124**	**14858**	**9212**	**6959**	**5830**
8				**15947**	**17384**	**16259**	**15532**	**9025**	**6594**	**5495**
9			**14537**	**16325**	**17284**	**15752**	**15080**	**7653**	**5078**	**4215**
in %										
0										
1										
2										**0.155**
3									**0.123**	**0.172**
4								**0.092**	**0.119**	**0.158**
5							**0.102**	**0.091**	**0.110**	**0.142**
6						**0.093**	**0.109**	**0.084**	**0.096**	**0.122**
7					**0.094**	**0.101**	**0.112**	**0.078**	**0.085**	**0.107**
8				**0.087**	**0.107**	**0.105**	**0.113**	**0.074**	**0.078**	**0.097**
9			**0.063**	**0.088**	**0.104**	**0.100**	**0.108**	**0.061**	**0.059**	**0.073**

Table 5.11: The estimated mean square error of predictions, I=10 and $\mu_i = 1000 \; \forall \; i$.

Total Outst. Payments			
Accident Year i	$\mathbf{TP}_{i,I-i}^{pred}$	$\mathbf{MSEP}^{1/2}$	**in%**
0	—	—	—
1	28334	7805	0.275
2	79993	12620	0.158
3	154695	17746	0.115
4	266583	23477	0.088
5	399602	28897	0.072
6	557377	34312	0.062
7	732939	39585	0.054
8	938439	45095	0.048
9	1187284	51005	0.043

Table 5.12: The estimated total outstanding payments and MSEP, I=10 and $\mu_i = 1000 \; \forall \; i$.

Chapter 6

Conclusion and Discussion

In non-life insurance, as the claims reserve is one of the largest portions of the company's liability, it is of great importance to study IBNR claims reserving problems using appropriate modeling approaches. Although there are well-used non-parametric methods such as Double Chain-Ladder (DCL) and Bornhuetter-Ferguson (BF) which can be easily interpreted and have a simple estimation process, we study in this project the parametric models under stochastic framework.

As the traditional autoregressive (AR) of order one model is not appropriate to model the claim counts, Kremer (1995) introduces an INAR(1) model for IBNR claim counts, and Bai (2016) extends Kremer's idea to propose a Poisson INAR(1) model for the unclosed claim counts. In this project, we make use of the ideas and further present a compound Poisson INAR(1) model for closed claim payments in which a mixed gamma distribution for individual claim sizes is assumed. The properties for the number of closed claims and the individual payments are studied. The compound model for the incremental aggregate claims is found to be over-dispersed which is one of the desired model properties, and it can be viewed as a generalization of Tweedie's model as it reduces to Tweedie's model when the closed rate is equal to 1. We apply the MLE technique to estimate the model parameters. However, as estimators for the model parameters using MLE technique are not expressed in closed forms, an algorithm is proposed to obtain parameter estimations using the system of estimating equations with an iterative process.

We conduct the simulation study to illustrate the model parameter estimations and prediction process. Comparing to the Tweedie's model studied in Wüthrich (2003), the MLE procedure for our model is much more complicated; however, we find that the algorithm proposed in this project for MLEs works efficiently. Throughout the simulation study, we find that the estimations for parameters can have a little improvement by having a larger size of the development triangle because in this case there are more observations to be used for estimations. From the practical point of view, using more than $I = 10$ may not be appropriate as the pattern of observations may change over a longer time period.

Using a sample of observations, we also provide numerical results to examine the impact of the size of loss triangle and overall mean of number of closed claims in each cell to the predictions. There are not significant improvements observed in terms of MSEP (in percentage) by increasing the size of the loss triangle. However, when the size of loss triangle is relatively large we observe that MSEP (in percentage) have a non-monotone pattern over development periods because there might be combined impacts from the estimation process as well as the number of observations available for different cells.

On the other hand, when the overall mean of number of closed claims is bigger we find that the prediction errors are reduced significantly as expected, since in this case more information is available to be used for parameter estimations. Although the mean square error of prediction is a little underestimated as we only consider the prediction error term of MSEP to estimate MSEP, the prediction results should give the reasonable range of estimated MSEP.

For the future research, we can treat the inflation rate as an unknown parameter and consider different impacts for accident years and development years to be more practical. Moreover, the complex claims which involve a litigation process or a serious medical review often require a longer time to get settled. For such cases, they also escalate the cost of claims. Thus, we can study the model that incorporates dependence between claim sizes and the frequency of claims.

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Appendix A

Tables for estimations of parameters

	I=6 μ_i=200	I=10 μ_i=200	I=10 μ_i=1000
$\hat{\rho}$	0.5294	0.3622	0.6445
$\hat{\mu}_0$	203.9509	212.3315	1024.1422
$\hat{\mu}_1$	203.9921	211.2775	1024.8246
$\hat{\mu}_2$	204.0609	213.2885	1026.597
$\hat{\mu}_3$	202.8100	211.0161	1024.5428
$\hat{\mu}_4$	203.9892	211.9057	1024.3622
$\hat{\mu}_5$	204.2655	212.0163	1024.1228
$\hat{\mu}_6$	203.8448	212.1536	1025.2298
$\hat{\mu}_7$	—	212.1835	1026.3905
$\hat{\mu}_8$	—	212.7131	1028.865
$\hat{\mu}_9$	—	213.0935	1019.1254
$\hat{\gamma}_0$	0.5212	0.2217	0.4114
$\hat{\gamma}_1$	0.1910	0.1778	0.2146
$\hat{\gamma}_2$	0.0900	0.1059	0.0664
$\hat{\gamma}_3$	0.0861	0.1040	0.0825
$\hat{\gamma}_4$	0.0406	0.0976	0.0891
$\hat{\gamma}_5$	0.0219	0.0807	0.0652
$\hat{\gamma}_6$	0.0493	0.0703	0.0572
$\hat{\gamma}_7$	—	0.0355	0.0065
$\hat{\gamma}_8$	—	0.0210	0.0001
$\hat{\gamma}_9$	—	0.0092	0.0001

Table A.1: Estimations of parameters of $R_{i,j}$

	I=6 μ_i=200	I=10 μ_i=200	I=10 μ_i=1000
$\hat{\alpha}$	1.9951	2.0069	2.0231
$\hat{\tau}$	1000.668	1000.612	1000.092
—	—	—	—
$\hat{q}_1$	0.0097	0.0103	0.0100
$\hat{q}_2$	0.0100	0.0100	0.0101
$\hat{q}_3$	0.0097	0.0101	0.0100
$\hat{q}_4$	0.0099	0.0092	0.0097
$\hat{q}_5$	0.0109	0.0102	0.0101
$\hat{q}_6$	0.0105	0.0103	0.0102
$\hat{q}_7$	—	0.0096	0.0102
$\hat{q}_8$	—	0.0095	0.0101
$\hat{q}_9$	—	0.0109	0.0105
$\hat{q}_{10}$	—	0.0073	0.0096

Table A.2: Estimations of parameters of $X_{i,j}^{(l)}$