Stochastic Modelling and Comparison of Two Pension Plans

by

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Abstract

In this project, we simulate the operation of a stylized jointly sponsored pension plan (JSPP) and a stylized defined contribution (DC) plan with identical contribution patterns using a vector autoregressive model for key economic variables. The performance of the two plans is evaluated by comparing the distribution of pension ratios for a specific cohort of new entrants. We find that the DC plan outperforms the JSPP in terms of expected pension ratio, and experiences only a moderate degree of downside risk. This downside risk is not enough to outweigh the upside potential even for a relatively risk-averse member, as reflected in the expected discounted utility of benefits under the two plans. Under more sophisticated rate stabilization techniques, the probability that the DC plan outperforms the JSPP increases. When the bond yield and stock return processes begin from values far above their long-term means (not far below, as is the case today), the DC plan is projected to outperform the JSPP even more frequently, because the higher required contributions accrue to the advantage of the individual member only, instead of also financing benefits for others.

Keywords: Pension Plan Comparison; Jointly Sponsored Pension Plan; Stochastic Simulation; Vector Autoregressive Model; Pension Ratio; Expected Discounted Utility.

Dedication

To my parents!

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Chapter 1

Introduction

1.1 Background and Motivation

Over the past few decades, occupational pension coverage has shifted from traditional defined benefit (DB) plans towards defined contribution (DC) pension plans in many countries. Factors contributing to this shift include increasing labor mobility, sustained problems with DB underfunding, and regulatory changes (Broadbent et al., 2006). For voluntary employer pensions, the shift towards DC pension plans was most pronounced in the U.S.. In Australia, where occupational pensions are mandatory, DC plans now cover most of the workforce. Canada was less affected by the shift, with most large public sector plans remaining DB. However, even in the public sector "some interest groups are pressing hard to convert their pension plans from DB to DC" (Brown and McInnes, 2014, p. 3).

DC plans provide portability and investment flexibility during the accumulation phase, but leave members with significant uncertainties in the decumulation phase. In most modern DC plans, members are responsible for making their own investment decisions. Whether these plans can provide an adequate income in retirement is thus linked to members' financial literacy, which is generally lacking. According to Brown and McInnes (2014), "while successful at vastly increasing DC pension coverage, the Australian program has been less successful at reducing poverty in seniors and displays on a large scale the problems associated with individually controlled savings plans" (p. 20). In the U.S., average pensions from DC plans fall short of the average benefits payable under DB plans.

Financial markets have also changed greatly. DC plan members with large allocations to equities may have benefited from the sharp rise in equity prices in the mid 1990s, while those who joined more recently have seen smaller rewards and greater volatility. A decline in long-term interest rates has also reduced the guaranteed income that the DC account balance could secure upon retirement.

As a result, there is growing interest among DC plan members in collective pensions to pool risks, and Simon Fraser University (SFU) is one example. SFU has maintained a

DC plan for faculty members for over 40 years. Now that the plan is mature and members are beginning to retire in significant numbers, they are finding that the benefits provided are inadequate. To provide faculty members with more predictable benefits at a better price, the SFU Faculty Association is considering switching from the current DC plan to the B.C. College Pension Plan, a jointly sponsored pension plan (JSPP). According to Kristjanson and Darrach (2012), a "JSPP is a contributory, defined benefit pension plan in which all contributing stakeholders and plan members have decision making and funding responsibility" (p. 2). JSPPs are unique in their funding structures, where employees and employers both have potentially unlimited risk and share it equally. Most JSPPs are governed by boards of trustees or directors, and are frequently used by hospital associations and public sector unions in Canada.

As suggested in the report prepared by PBI Actuarial Consultants Ltd. (2015), the College Pension Plan is able to mitigate risks, including investment risk, inflation risk and longevity risk, and provide a more certain retirement income. It also has the advantage of low management and administration expenses. The report contains a number of comparisons that approximate the retirement income members could receive depending on their current age and retirement age under various return scenarios. In most scenarios, the College Pension Plan tends to outperform the current plan. Most members, except for those with strong financial literacy skills, would be better off in the College Pension Plan that removes much of the investment risk and provides a predictable pension benefit.

One limitation of the PBI report is that all assumptions are deterministic. Without accounting for the volatility in future salary increases, inflation, investment returns and annuity purchase rates, the comparison lacks an important dimension: risk. This affects both options: in terms of benefit volatility under the current DC plan, and contribution volatility under the College Pension Plan. Our motivation is to extend the work done by PBI to a stochastic context and compare the value of the two plan options to a particular cohort of new entrants. We focus on new entrants exclusively because, if SFU were to join the College Pension Plan, enrollment would be mandatory for new employees, while existing faculty members would have the option to stay in the current DC plan.

In order to stochastically simulate the operation of the two plans, we need an asset model to generate future economic scenarios. In dealing with economic variables, often the value of one variable is not only related to its predecessors in time, but also depends on past values of other variables. Consequently, we choose a vector autoregressive (VAR) model, which assumes a linear relationship between multiple economic variables, and predicts future values based on linear functions of past observations. In a VAR model, there are n state variables and n equations to express the relationship between each variable and its own lagged values, as well as current and past values of the remaining $n-1$ variables.

A reasonable performance criterion to evaluate alternative pension plans with comparable contribution levels is the pension ratio: the ratio of benefits under one plan to benefits under the other. We use the value-at-risk (VaR) of the pension ratio to assess how often the SFU plan can deliver the same or better benefits as the College Pension Plan. In addition, we use expected discounted utility to perform a welfare comparison of benefits under the two schemes. Expected utility theory has been used as a major paradigm in decision making problems (Schoemaker, 1982). It serves as the second performance criterion in our study, and provides additional evidence about whether the representative cohort would benefit from joining the College Pension Plan.

1.2 Literature Review

Most of the actuarial literature relating to quantitative comparisons of alternative pension plans focus on pure DB and DC plans. Samwick and Skinner (1998) used a detailed survey of pension formulas in the Survey of Consumer Finance to estimate the average pension benefit for a sample of both plans. They found that DC plans could strengthen the financial security of retirees, and their conclusion was robust to a number of specifications. However, the paper was written in a period when stocks provided high returns. Also, some key variables used in the analysis, such as annuity purchase rate, were fixed and did not anticipate the downward trend in long term interest rates. Their findings might be very different today.

Blake, Cairns and Dowd (2001) investigated a range of stochastic asset return models and asset allocation strategies, to estimate the distribution of future pension ratios (i.e., ratios of DC pension to DB pension). They explored the dynamics of interest rates, earnings, unemployment and asset allocation. The application of the well-established risk measure, value-at-risk, provided a simple and practical-to-implement methodology to evaluate alternative pension plans. The conclusion was that DC plans can be extremely risky relative to a DB benchmark. Value-at-risk estimates were most sensitive to the choice of asset-allocation strategy, and less sensitive to the choice of asset model. Asset models used in this paper included the Wilkie (1986) model, which was the first stochastic model for use by actuaries that incorporated a cascade structure, where each variable depends only on prior values of that variable and the values of variables that lie above them on the cascade structure. Under this structure, once a variable is appropriately calibrated, the calibration of subsequent variables lower on the cascade structure will have no impact on the previously calibrated variables.

Blake, Cairns and Dowd (2003) compared alternative decumulation strategies, including a conventional life annuity, an equity-linked annuity, and an equity-linked income distribution programme. To measure the performance of different strategies, they calculated the plan member's expected discounted lifetime utility. This framework captured an individual's attitude towards risk, and allowed the authors to optimize asset portfolios by maximizing the utility function. They concluded that the optimal choice of distribution programme was fairly insensitive to a member's risk-aversion level, but was greatly affected by equity proportions.

McCarthy (2003) employed a stochastic life-cycle model to compare the value (utility) provided by DB plans and DC plans to a risk-averse individual under various economic conditions. The analysis showed that, in the presence of both wage and asset risks, a DC plan would provide better welfare for younger individuals because, over their long horizon, the benefit of higher equity exposure combined with a reasonable equity risk premium would outweigh the value of guaranteed wage-indexed benefits. By contrast, DB plans would be more attractive to older workers due to the workers' shorter horizon, their relatively low capacity for bearing financial risk, and the lower cost of annuities provided through DB plans.

In recent years, research has shifted to the comparison of individual DC plans to modern hybrid plans and explored the effectiveness of intergenerational risk sharing in improving DC outcomes. Here, intergenerational risk sharing means that different "generations of plan members enter into hedging contracts for all or part of the mortality, investment, inflation, and/or other risks among themselves" (CIA, 2015).

Cui et al. (2011) performed a welfare comparison between collective plans and the optimal individual pension scheme without risk sharing. They concluded that the individual DC scheme was not the optimal funded scheme, even in a frictionless world with sophisticated individuals. Collective funded schemes could be a positive-sum game from a welfare perspective. However, even well-structured intergenerational risk sharing was a zero-sum game in market value terms, suggesting that welfare improvements for one cohort did not come at the expense of another cohort.

Bovenberg et al. (2007) discussed how collective pension schemes may help to relieve some of the market incompleteness that arose from various constraints to optimal individual decision making. The advantage of these schemes was to allow for risk sharing with future generations. On the other hand, the disadvantage was the existence of intertemporal smoothing as shocks might not be spread out over the individual's remaining lifetime.

Gollier (2008) derived operational rules and optimal investment strategies for a collective DC plan to optimize the sharing of risk across generations. From the market value perspective, the distributed benefit was very sensitive to the financial situation of the fund when solvency was an issue. It meant that intergenerational risk-sharing did not work well in these circumstances. The estimation of the welfare gain also showed that better intergenerational risk-sharing did not reduce the risk borne by each generation. The additional collective risk exposure was only partially offset by pooling and diversification.

Bloomestein et al. (2009) analyzed the trade-off between the uncertainty in contributions and benefits embedded in a range of designs. The funding ratio (ratio of assets to liabilities) and the replacement rate (ratio of benefits to salaries) were key criteria for evaluating the risk sharing characteristics. The stochastic simulations showed that hybrid plans appeared to be more efficient and sustainable forms of risk sharing, compared to traditional DB or DC plans. However, the choice of a specific pension arrangement depends on the preferences of plan members, and in particular their degree of risk aversion and their ability to commit to an intergenerational risk sharing contract.

Hoevenaars and Ponds (2008) conducted a value-based asset liability management analysis which revealed value transfers and implicit payoffs between generations. Moving away from the expected utility framework, they identified embedded options in the pension deal and compared the market value of these options for different cohorts of members. The economic modelling framework adopted a pricing kernel and defined an affine term structure of interest rates, as in Cochrane and Piazzesi (2005). The conclusion regarding value transfer was that a reallocation of risk bearing from flexible contributions to flexible benefits led to value redistribution from old to young.

The method in Hoevenaars and Ponds (2008) could also be used to estimate and compare risk-adjusted values of the net benefits provided under alternative designs. Lekniute at al. (2014) did exactly that, applying the value-based approach to investigate the market consistent value of benefits projected to be received by different generations of members among the U.S. state pension plans subject to a number of plan options. The market-based valuation demonstrated that the current pension contract implied net benefits, causing substantial financial burden to all cohorts of future taxpayers, which was the reason that alternative contracts were considered.

1.3 Outline

The paper is organized as follows: Chapter 2 sets up the VAR model and describes the economic scenarios generated. Chapter 3 describes the College Pension Plan and the design of a stylized JSPP. We simulate the operation of the stylized JSPP, and use the results to project cash flows for a representative cohort of new entrants. Chapter 4 describes the SFU plan and the design of a stylized DC plan. Cash flow projections are made for the same cohort as in the previous chapter. In Chapter 5, we analyze the performance of the stylized JSPP and the stylized DC plan by looking at the pension ratio and expected utility. In Chapter 6, we introduce the market-value based performance criterion which allows us to compare pension plans whose contribution patterns are different. Our conclusions and potential future work are summarized in Chapter 7.

Chapter 2

Economic Scenario Generator

2.1 The VAR Model

As in Heidelberg (2005), we model the return dynamics by a first-order VAR model,

$$z_{t+1} = \Phi z_t + P\epsilon_{t+1} \tag{2.1}$$

where z_t is a (5×1) vector of centered state variables and $\epsilon_{t+1} \overset{i.i.d}{\sim} N(0, I)$ is a (5×1) vector of innovations. More precisely,

$$z_t = x_t - \mu \tag{2.2}$$

where x_t is the (5×1) vector of original state variables and μ is the vector of their historical means. By subtracting μ, we rule out the intercept term in the VAR model, as well as estimation inaccuracy on it. Φ and P are both (5×5) matrices. Φ contains the autoregressive coefficients of the VAR model and P is the Cholesky decomposition of the covariance matrix Σ for residuals. In other words, P is a lower triangular matrix and satisfies

$$PP^T = \Sigma. \tag{2.3}$$

2.2 Data and Parameter Estimation

The state variables that enter the VAR model include price inflation, 1-month interest rate, 10-year zero-coupon bond rate, stock return from the Toronto Stock Exchange (TSX) index, and stock return from the Standard & Poor's 500 (S&P 500) index. We express the S&P 500 index in Canadian dollars to rule out fluctuations in currency exchange rates, and use total stock returns (that is, including dividend yield). Historical data are monthly continuously compounded values obtained at monthly frequency for the period 1991:05 to 2016:04. Using monthly frequency has the advantage of providing more data points and capturing the volatility of the data. The reason for excluding pre-1991 records is that the

Bank of Canada adopted an inflation-control target in 1991, which "aims to keep total CPI inflation at the 2 per cent midpoint of a target range of 1 to 3 per cent over the medium term" (Bank of Canada, n.d.). Data is available from the following sources:

- Values of the consumer price index with base year 2002, considering all of Canada and not excluding any items are retrieved from CANSIM table 326-0020. The force of monthly inflation in month t is defined as:

$$\tilde{\lambda}_t = ln\frac{CPI_t}{CPI_{t-1}}, \tag{2.4}$$

  where CPI_t is the value of the index at the end of month t.

- The yield on 1-month Canadian treasury bills in month t (i_t^1) is retrieved from CANSIM table 176-0043, where it is quoted as an annual effective rate. The corresponding monthly force of interest on 1-month treasury bills is defined as:

$$\tilde{y}_t^1 = \frac{1}{12}ln(1+i_t^1). \tag{2.5}$$

- The yield curves for zero-coupon bonds with terms to maturity ranging from 3 months to 30 years are available from the Bank of Canada on a daily basis. The 10-year bond yield observed on the first trading day of month t (i_t^{120}) is used as a proxy for the long-term interest rate. The 10-year bond yield at the beginning of month t, expressed as the force of interest, is defined as:

$$\tilde{y}_t^{120} = \frac{1}{12}ln(1+i_t^{120}). \tag{2.6}$$

- For Canadian equity returns, the monthly S&P/TSX composite index values are from CANSIM table 176-0047 (2000=1000). This index includes the equity prices of the largest companies on the Toronto Stock Exchange and their dividend yields. The continuously compounded monthly total return on Canadian equities during month t is defined as:

$$\tilde{\pi}_t^C = ln\Big[\frac{CEI_t}{CEI_{t-1}} + \frac{CDY_t}{12}\Big], \tag{2.7}$$

  where CEI_t is the value of the S&P/TSX equity index at the end of month t and CDY_t is the annual Canadian dividend yield in respect of month t.

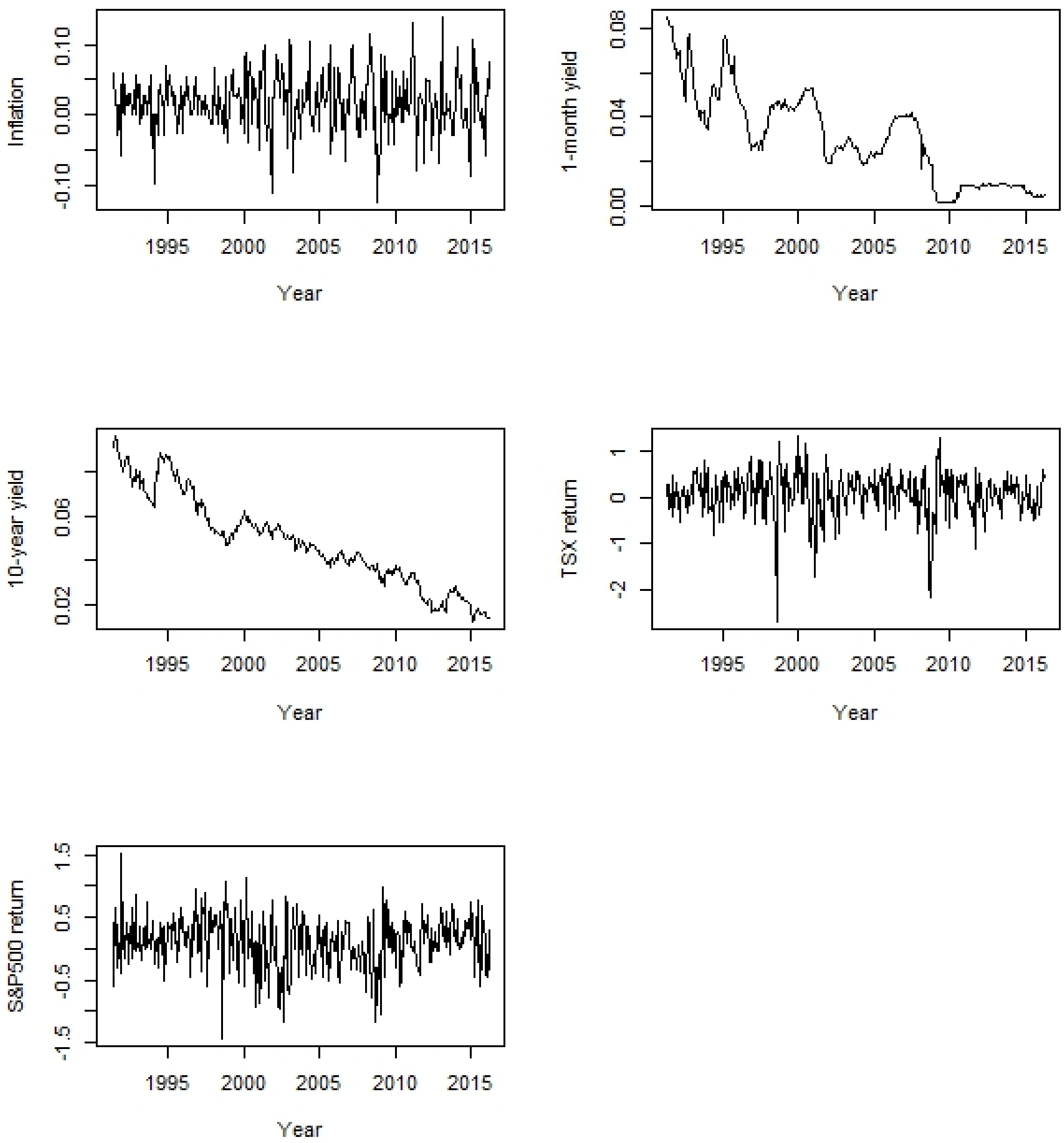
- For U.S. equity returns, values of the monthly S&P 500 composite index are available on the Bloomberg database. This index includes 500 large companies with common stock listed on the NYSE or NASDAQ, and dividend yields reflect the ordinary cash dividend paid by these companies over an annual accrual period. We defined the continuously compounded monthly total return on U.S. equities during month t as:

$$\tilde{\pi}_t^U = ln\Big[\frac{SEI_t}{SEI_{t-1}} + \frac{SDY_t}{12}\Big], \tag{2.8}$$

where SEI_t is the value of the S&P 500 equity index at the end of month t and SDY_t is the annual dividend yield on the S&P 500 in respect of month t.

Figure 2.1 shows historical data for these five variables converted to annual scale. While bond yields depend greatly on past values and suggest strong auto-correlation, there is no significant pattern for inflation rates and equity returns. Current interest rates are at a historically low level. Summary statistics of historical data can be found in Table 2.1 panel (a).

Figure 2.1: Historical data of the VAR model



Panel (b) and (c) in Table 2.1 show estimates of Φ and P with p-values in parentheses, obtained by using the R package *vars*. Inflation appears to be weakly related to Canadian equity returns besides its own lag, but the corresponding R^2 (0.0551) is very low. Stock returns have even less relationship to lagged variables; with an R^2 of 0.0335 and 0.0027, and no significant autocorrelations, these returns more or less follow white noise processes with highly correlated innovations. Panel (c) also confirms that stock returns have the highest volatility. Interest rates are mostly explained by their own lagged values, and the volatility of short term yield is higher than the volatility of the long-term yield.

When we fit the VAR model, an important assumption is that the process is stationary, that is, its statistical properties such as mean and autocovariances are fixed and do not change over time. Stationarity is crucial for being able to describe the stochastic behavior by the simple VAR model and to estimate the parameters. As introduced in Heidelberg (2005), the stationarity condition for a VAR(1) model requires all eigenvalues of Φ have modulus less than 1. Here the absolute values of the eigenvalues are 0.9892, 0.9317, 0.1453, 0.1453 and 0.0266. Since they are all smaller then one, the stationarity condition is satisfied.

Table 2.1: Summary statistics and VAR estimation results

a) Summary statistics	$\tilde{\lambda}_t$	$\tilde{y}_t^1$	$\tilde{y}_t^{120}$	$\tilde{\pi}_t^C$	$\tilde{\pi}_t^U$	
μ	0.0015	0.0026	0.0040	0.0066	0.0076	
σ	0.0034	0.0017	0.0017	0.0420	0.0365	
b) VAR estimates (Φ)	$\tilde{\lambda}_t$	$\tilde{y}_t^1$	$\tilde{y}_t^{120}$	$\tilde{\pi}_t^C$	$\tilde{\pi}_t^U$	R^2
$\tilde{\lambda}_{t+1}$	0.1415	0.1116	-0.1101	0.0157	-0.0107	0.0551
	(0.0154)	(0.6183)	(0.6206)	(0.0066)	(0.1017)	
$\tilde{y}_{t+1}^1$	-0.0005	0.9510	0.0300	-0.0001	0.0001	0.9724
	(0.9200)	(0.0000)	(0.1170)	(0.8730)	(0.8750)	
$\tilde{y}_{t+1}^{120}$	0.0009	0.0239	0.9707	-0.0002	-0.0001	0.9876
	(0.7795)	(0.0675)	(0.0000)	(0.5683)	(0.7798)	
$\tilde{\pi}_{t+1}^C$	-0.1494	-2.9637	3.2773	0.1226	0.0725	0.0335
	(0.8383)	(0.2938)	(0.2427)	(0.0911)	(0.3809)	
$\tilde{\pi}_{t+1}^U$	-0.0491	-0.7141	1.4017	-0.0038	0.0328	0.0027
	(0.9390)	(0.7740)	(0.5710)	(0.9520)	(0.6530)	
c) VAR estimates (P)	$\tilde{\lambda}_t$	$\tilde{y}_t^1$	$\tilde{y}_t^{120}$	$\tilde{\pi}_t^C$	$\tilde{\pi}_t^U$	
$\tilde{\lambda}_t$	0.0033					
$\tilde{y}_t^1$	0.0000	0.0003				
$\tilde{y}_t^{120}$	0.0000	0.0000	0.0002			
$\tilde{\pi}_t^C$	0.0051	-0.0022	-0.0033	0.0411		
$\tilde{\pi}_t^U$	-0.0025	0.0027	-0.0021	0.0219	0.0292	

2.3 Simulation Results

We can generate future economic scenarios by forward iterating the VAR model in equation (2.1). By assuming normally distributed innovations, we draw stochastic disturbances from random number generators. In this way, historical correlations and current market shocks are taken into account, and the values of the variables for the next period are computed. The scenario set *nsim* (where *nsim* goes from 1 to 10,000) of the five state variables x_{t+h}, for $h = 1, 2, ..., 720$, given x_t, can be created using:

$$\begin{aligned} z_{nsim,t+h} &= \Phi^h z_t + \sum_{i=0}^{h-1} \Phi^i P \epsilon_{nsim,t+h-i} \\ x_{nsim,t+h} &= z_{nsim,t+h} + \mu \end{aligned} \tag{2.9}$$

Monthly economic scenarios are generated for a future period of 60 years (i.e., monthly steps), starting in 2016:05. Our pension plan projections begin on January 1, 2017, so we discard the initial 8 iterations which relate to 2016. Annualized force of inflation and equity returns can be obtained directly from the simulations:

$$\begin{aligned} \lambda_t &= \sum_{h=12(t-1)+1}^{12t} \tilde{\lambda}_h, \\ \pi_t^C &= \sum_{h=12(t-1)+1}^{12t} \tilde{\pi}_h^C, \\ \pi_t^U &= \sum_{h=12(t-1)+1}^{12t} \tilde{\pi}_h^U, \end{aligned} \tag{2.10}$$

where $h = 1$ corresponds to the rates applicable to January 2017 and t is measured in years.

To calculate the return on bonds, we first take the continuously compounded yields on short (1-month) and long (120-month) bonds at the end of year t (month $12t$) obtained from equations (2.5) and (2.6) and express them as annual effective rates:

$$\begin{aligned} i_t^1 &= exp(12\tilde{y}_{12t}^1) - 1, \\ i_t^{120} &= exp(12\tilde{y}_{12t}^{120}) - 1. \end{aligned} \tag{2.11}$$

Next, we assume that the yield curve is linear between the 1-month and 10-year maturities. Mathematically, the 9-year spot rate expressed as an annual effective rate, i_t^{108}, and the corresponding continuously compounded spot rate, y_t^{108}, applicable at the end of year t are calculated as:

$$\begin{aligned} i_t^{108} &= \frac{i_t^{120} - i_t^1}{119} \times 107 + i_t^1, \\ y_t^{108} &= ln(1 + i_t^{108}). \end{aligned} \tag{2.12}$$

The continuously compounded 10-year spot rate applicable at the end of year t (month $12t$), expressed as an annual rate, is denoted by y_t^{120} and is obtained directly from the simulations:

$$y_t^{120} = 12\tilde{y}_{12t}^{120}. \tag{2.13}$$

Let $P^{(n)}$ denote the projected market price of an $n-$year zero-coupon bond at time t (measured in years), which can be obtained from the projected $n-$year spot rate. Annual returns on a constant maturity portfolio of 10-year zero-coupon bonds (π_t^B) during the period [t-1,t), assuming annual sale and repurchase, are calculated from the underlying bond prices:

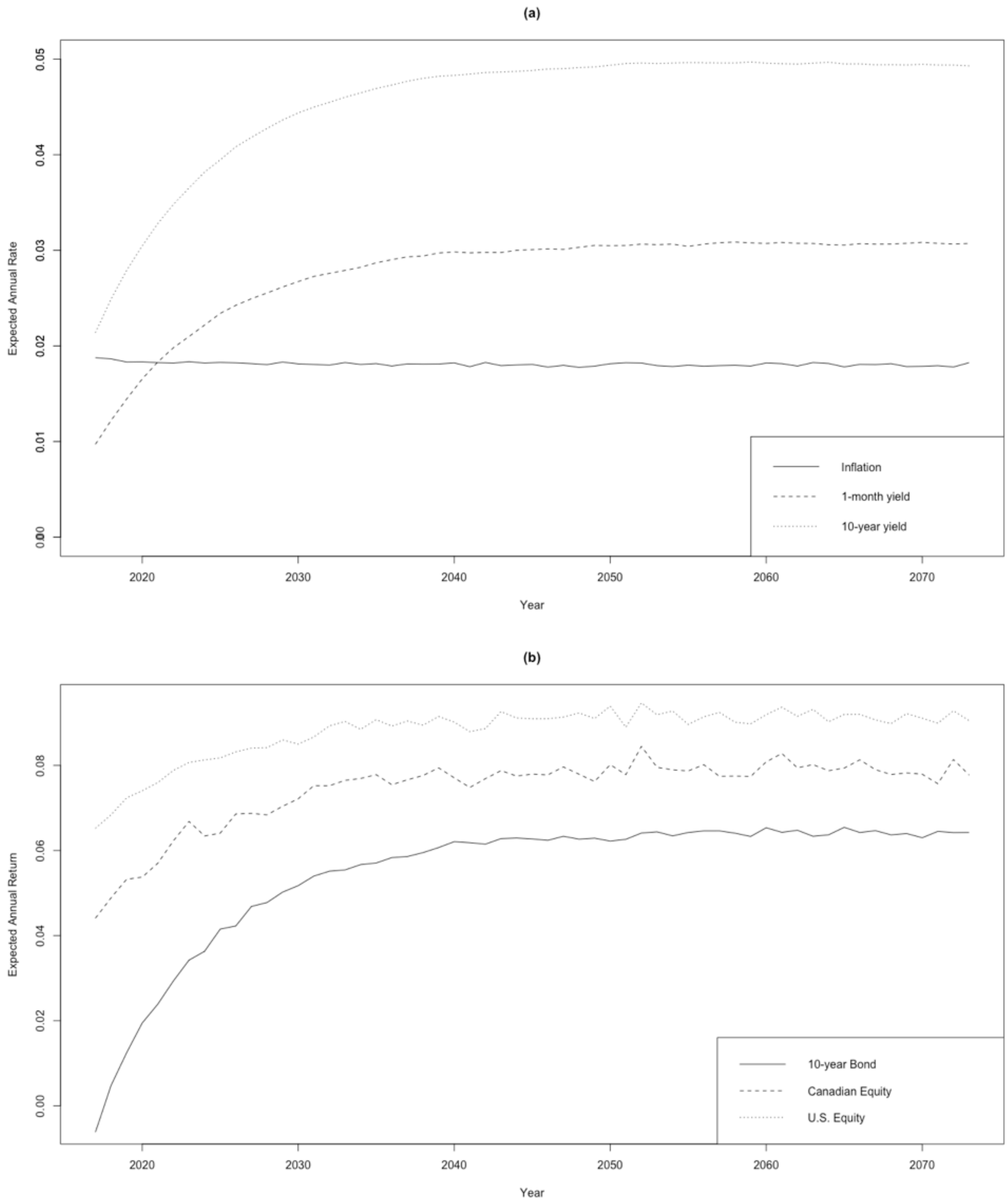
$$\begin{aligned} \pi_t^B &= ln(\frac{P_t^{(9)}}{P_{t-1}^{(10)}}) \\ &= p_t^{(9)} - p_{t-1}^{(10)}, \end{aligned} \tag{2.14}$$

where $p_t^{(n)}$ is the projected log price of an $n-$year bond. Combining equation (2.14) with (2.12) and (2.13), we have

$$\pi_t^B = 10y_{t-1}^{120} - 9y_t^{108}. \tag{2.15}$$

Figure 2.2 (a) shows the average of 10,000 simulated annual rates for inflation (λ_t), 1-month yield (y_t^1) and 10-year yield (y_t^{120}) each year. Expected inflation tends to its long term mean quickly, and then fluctuates in a small range. Interest rates are expected to go back to their mean in around 20-25 years, staying stable afterwards. Panel (b) shows the average of 10,000 simulated annual returns on the portfolio of 10-year bonds (π_t^B), Canadian equities (π_t^C) and U.S. equities (π_t^U). Stock markets have a higher average return than the bond, and the S&P 500 outperforms the other two securities.

Figure 2.2: Simulation results



Chapter 3

Simulation of Stylized Jointly Sponsored Pension Plan (JSPP)

In this chapter, we first introduce the College Pension Plan. Our goal is to simulate the operation of a plan similar to the College Pension Plan with certain simplifications. General assumptions and notation are presented in section (3.2), which are followed by features of the stylized JSPP in section (3.3). To investigate the impact of rate stabilization techniques, we describe two alternative designs in section (3.4).

3.1 BC College Pension Plan

British Columbia's public sector pension plans (BC plans) include the College Pension Plan, the Municipal Pension Plan, the Public Service Pension Plan and Teacher's Pension Plans (Municipal Pension Plan, n.d.). They are pre-funded so each generation pays in advance for its own pension benefits. Costs and risks are shared between employees and employers. A basic element of each of these plans is that guaranteed pensions are based on a DB formula using the member's pensionable service and salary. Another element is inflation protection. This is not a guaranteed benefit and is provided based on the availability of funds. Contributions may change depending on the funded status of the plan. Each plan uses the BC Investment Management Corporation as its investment agent, which provides sophisticated and low-cost investment management of the funds. The total cost of investment management and pension administration for the plans is about one quarter of one per cent.

The College Pension Plan, designed almost 50 year ago, is by far the smallest of the four BC's public sector pension plans. It maintains retirement benefits for around 25,000 senior administrators and faculty providing educational services at 23 BC colleges and universities. In 2000, the College Pension Plan shifted from government sponsorship to joint sponsorship and trusteeship. The plan is funded by employee and employer contributions, and under the new model risks are shared equally by the two parties. The joint trust agreements require that contribution rates and benefits be reviewed triennially based on an actuarial valuation. The features described below are from College Pension Plan 2015 Annual Report (College Pension Plan [CPP], 2016a), College Pension Plan Funding Policy (CPP, 2016b), and College Pension Plan Statement of Investment Policies and Procedures (CPP, 2016c).

1. Demographic Profile:
   Plan membership consists of 13,807 active members who are currently contributing (54% of the membership); 5,170 inactive plan members who have terminated their employment but left their benefits in the plan (20% of the membership) and 6,453 retired plan members who are receiving a pension, including a survivor or disability pension (26% of the membership).

2. Contributions:
   Both plan members and employers pay contributions to fund future pension benefits; plan members contribute through automatic deductions from their employment earnings. A portion of these contributions goes to the basic account, which covers members' basic pensions; another portion goes to the inflation adjustment account, which covers cost-of-living adjustments (COLAs). Table 3.1 is a summary of contribution rates as a percentage of salaries from the College Pension Plan 2015 Annual Report.

Table 3.1: College Pension Plan contribution rates

Effective Date	On salary up to YMPE[1]		On salary over YMPE	
	Member	Employer	Member	Employer
January 1, 2016	9.86%	9.96%	9.86%	9.96%
September 1, 2013	9.60%	9.70%	10.35%	10.45%

[1] YMPE = Year's Maximum Pensionable Earnings, the maximum earnings on which Canada Pension Plan contributions are made.

3. Asset Allocation and Investments:
   When members retire, their pension is funded by their own contributions, their employers' contributions and investment returns. Based on current assumptions, approximately 30 cents of every dollar a retired member receives come from contributions they made and their employer made; the remaining 70 cents come from investment returns. To achieve the objective of meeting the pension benefits promise, the Board has adopted the long term asset mix and allowable ranges as shown in Table 3.2. Diversifying investments is a sound way to balance investment risk while generating returns, especially in a global economy where turbulence is not uncommon. In the last three years, College Pension Plan investment portfolio earned 7.3% net of fees for

the fiscal year 2014/15, 17.5% for 2013/14, and 10.3% for 2012/13, which all exceeded the market benchmark.

Table 3.2: College Pension Plan asset mix: allowable ranges and long term policy

Asset Class	Allowable Range (%)		Long Term Policy Asset Mix(%)
	Minimum	Maximum	
Short Term	0	10	2
Mortgages	0	10	5
Nominal Bonds	5	17	10
Real Return Bonds	0	10	5
Fixed Income Sub-total	12	35	22
Canadian Equities	8	18	13
Global Equities	17	29	22
Emerging Markets	4	14	9
Public Equity Sub-total	35	55	44
Real Estate[1]	11	23	18
Real Estate Sub-total	11	23	18
Private Placements[1,2]	1	11	6
Infrastructure and Renewable Resources[1,2]	5	15	10
Sub-total	7	22	16
Other[3]	0	5	5

[1] Due to the illiquid nature of these assets, the upper limit may be exceeded on a temporary basis.
[2] Private Placements may be either debt or equity.
[3] "Other" includes strategies or investments specifically approved by the Board that do not correspond to the listed asset classes.

4. Basic Pension:
   The College Pension Plan provides members or their beneficiaries with a basic lifetime pension benefit based on highest average salary and years of service. Here "highest average salary" means the average annual salary earned by a member during the 5 years of pensionable service in which the salaries were highest. Normal retirement age is 65 for all members. The unreduced guaranteed benefit is calculated in the form of a single life annuity guaranteed for 10 years:

   $$2\% \times \text{five-year highest average salary} \times \text{total pensionable service (years)}.$$

5. Cost-of-living Adjustments (COLA):
   Cost-of-living Adjustments to pensions in pay are managed through a separate inflation adjustment account. Future increases are not guaranteed; however, once granted, COLA becomes part of the members' basic lifetime pension. On January 1, 2015, retired members received a COLA of 1.83 per cent. The COLA cannot exceed the

change in the consumer price index or the inflation adjustment cap set every three years after each valuation.

6. Actuarial Assessment:

   Asset values and investment returns are smoothed over a five-year period. The smoothed value of the assets is limited to no more than 108% and no less than 92% of the market value of the assets.

   The entry-age normal (EAN) cost method is used as the funding basis. Triennial valuations take a long-term perspective and assumptions are based on best estimates with margins for adverse deviations. The plan is required to comply with the following going-concern valuation requirements:

   (1) pay normal cost

   (2) if there is an unfunded liability, it should be amortized over 15 years

   (3) if there is a surplus, the surplus allocation policy is:

   - firstly, stabilize the contribution rate by adjusting, or establishing, a rate stabilization reserve,
   - secondly, transfer surplus to the inflation adjustment account to support the indexing at the current maximum sustainable level,
   - thirdly, consider benefit improvements or contribution rate reductions.

   The College Pension Plan is exempt from the solvency funding requirements of the BC pension legislation. Key financial statistics arising out of previous actuarial valuations are summarized in Table 3.3.

Table 3.3: College Pension Plan key financial statistics for the last two valuations

Valuation Date	Assets (Basic Account) (000's)	Actuarial Liabilities (000's)	Funded Ratio (Going Concern)
August 31,2015	$4,361,338	$4,294,246	101.6%
August 31,2012	$3,505,245	$3,618,924	97.0%

3.2 General Assumptions and Notation

In our model, we make some important assumptions.

1. Contributions are received and benefits are paid at the beginning of the year.
2. Expenses related to management and administration are ignored.
3. The source of contributions (employer vs. member) is irrelevant. All contributions are considered together.
4. Retirement age is 63 for males and 62 for females. There are no early retirements or pre-retirement decrements.
5. Actual salary increases (s_t) are equal to inflation plus an additional 0.75% per year, that is
$$s_t = \lambda_t + 0.0075. \tag{3.1}$$
6. After retirement, pensions are payable for a term certain which is equal to life expectancy at retirement (24 years for male and 27 for female members).

Assumption 6 is based on the results of the Canadian registered pension plan study conducted by the Canadian Institute of Actuaries (CIA, 2014). Base mortality follows the 2014 Public Sector Mortality Table (CPM2014Publ). We combine the base table with CPM Improvement Scale B (CPM-B), which contains improvement rates by age that decrease in a linear fashion for years 2012-2030 and ultimate rates applicable for all years after 2030. The average number of years that a person age x is expected to live is:

$$e_x = \sum_{k=1}^{100} {}_kp_x \tag{3.2}$$

where ${}_kp_x$ is the probability that (x) survives k years. Table 3.4 lists some of these life expectancies.

Table 3.4: Canadian life expectancy for selected ages

Age (x)	61	62	63	64	65	66	67
e_x (Male)	25.5	24.6	23.7	22.8	21.9	21.0	20.1
e_x (Female)	27.5	26.5	25.6	24.7	23.8	22.9	22.0

Next, we introduce some notation. We let:

- e be the entry age,
- r be the retirement age,
- l be the life expectancy at time of retirement,
- r_t^P be the annual portfolio return during the period $[t-1, t)$,
- s_t be the actual salary increase rate at time t,
- $\ddot{a}_{\overline{h|}i}$ be the present value of an $h-$payment annuity certain payable in advance using an effective rate of interest of i.

The following variables are also of interest, where we use a subscript of $\{x, t, k\}$ to denote the value for a person age x at time t who entered the plan in the k^{th} valuation period. Furthermore, we add an asterisk to represent a corresponding projected value, whenever necessary. For example,

- $Sal_{x,t,k}$ is the actual annual salary; $Sal^*_{x,t,k}$ is the projected annual salary based on the most recent valuation;
- $C_{x,t,k}$ is the annual contribution;
- $B_{x,t,k}$ is the annual benefit payable to a pensioner age x based on actual salaries; $B^*_{x,t,k}$ is the corresponding annual benefit based on projected salaries.

3.3 Stylized JSPP

Now we introduce a stylized JSPP based on the College Pension Plan. For simplicity, we consider only the guaranteed benefits payable from the Basic Account and completely ignore cost-of-living adjustments and the Inflation Adjustment Account. This makes our JSPP very similar to a DB plan, where benefits are guaranteed and only contributions fluctuate. However, unlike most studies comparing traditional DB and DC plans, we explicitly take into account the variability of contributions under the JSPP since they do affect the net value of the benefits received.

At retirement, pension payments are determined based on the members' salary and years of service as follows:

$$B_{x,t,k} = 2\% \times (r - e) \times FAS_{x,t,k},\ x > r \tag{3.3}$$

where $FAS_{x,t,k}$ is the average annual salary earned by a member during the last 5 years of pensionable service:

$$\begin{aligned} FAS_{x,t,k} = & \frac{1}{5}(Sal_{r-1,r-e+3k-1,k} + Sal_{r-2,r-e+3k-2,k} + Sal_{r-3,r-e+3k-3,k} \\ & + Sal_{r-4,r-e+3k-4,k} + Sal_{r-5,r-e+3k-5,k}), \end{aligned} \tag{3.4}$$

and the age at entry, e, is equal to $x - (t - 3k)$.

Note that we have replaced highest average salary with final average salary. This is a reasonable simplification, since 97% of the simulated salary increase rates are positive and the minimum value is no lower than -4%, so final average salary is almost identical to the highest average salary used by the College Pension Plan. Total benefit payments made from the plan at time t are

$$B_t = \sum_x \sum_k B_{x,t,k}. \tag{3.5}$$

We conduct triennial valuations, starting January 1, 2017, to review the adequacy of plan funds and revise contribution rates. Contribution rates (c_t) are determined at each valuation and used to determine actual annual contributions on behalf of each member:

$$C_{x,t,k} = c_t \times Sal_{x,t,k}. \tag{3.6}$$

The total contributions made to the plan at time t are

$$C_t = \sum_x \sum_k C_{x,t,k}. \tag{3.7}$$

3.3.1 Membership

In the College Pension Plan Actuarial Valuation Report (Eckler, 2016), membership information is displayed in 5-year age intervals. In order to make triennial valuations easier, we regroup the membership data into 3-year age intervals assuming a uniform distribution of members within each original 5-year age band. Since the liabilities for inactive members are only 4.7% of the total liabilities, we ignore the whole inactive group. Membership data used in our valuation as at January 1, 2017 are summarized in Appendix A.

Plan membership is projected forward for future actuarial valuations. Under our assumptions, existing members leave the plan only by reaching their life expectancy. On each valuation date, the same number of new entrants with the same age distribution (shown in Table A.1) join the plan. As a result, plan membership is not stationary during our projection horizon.

3.3.2 Assets

The asset allocation is designed to reflect the characteristics of the College Pension Plan using the state variables available in the VAR model. As Table 3.2 suggests, real estate, private placements, infrastructure and renewable resources are considered illiquid, thus we only include fixed income and public equity in the asset portfolio. Note that we use an index to model equity returns while the actual fund adopts active management which has outperformed the index. In this case, our assumption is more conservative.

Since short-term bonds only make up a small percentage of the portfolio, we let the entire fixed income allocation of the stylized JSPP consist of 10-year bonds only, with a weight of 35% of the total portfolio ($\approx \frac{22\%}{22\%+44\%}$). Canadian equities are 13/44 of public equities, equivalently 20% of our portfolio. The remaining 45% are modelled as U.S. equities, amalgamating global equities and emerging markets into this category. We assume that the asset mix remains the same throughout the projection horizon. Re-balancing is done at year end, right before contributions are made, to make sure that the target asset allocation is maintained. The maturity of the bond portfolio is reset every year by selling all remaining (now 9-year maturity) bonds and purchasing new 10-year bonds at prevailing market prices.

The simulation of the VAR model provides the annualized forces of interest for the Canadian equity returns (π_t^C) and the U.S. equity returns (π_t^U). We first transfer them into annual effective rates, respectively:

$$\begin{aligned} r_t^C &= exp(\pi_t^C) - 1, \\ r_t^U &= exp(\pi_t^U) - 1, \\ r_t^B &= exp(\pi_t^B) - 1. \end{aligned} \tag{3.8}$$

The annual portfolio return r_t^P on $[t-1, t)$ is then calculated as:

$$r_t^P = 0.35 r_t^B + 0.2 r_t^C + 0.45 r_t^U. \tag{3.9}$$

The market value of plan assets is determined recursively, assuming contributions and benefit payments are made at the beginning of the year:

$$MV_t = (MV_{t-1} + C_{t-1} - B_{t-1})(1 + r_t^P). \tag{3.10}$$

The actuarial value of plan assets (AV_t) is taken as the 5-year smoothed market value subject to the same corridors as under the College Pension Plan. Details of the smoothing technique are in Appendix B.

3.3.3 Valuation Assumptions

The assumptions we need for each valuation are future valuation rates and future salary increases. These vary at each time point under each scenario. We set y_t^V, the funding valuation rate at time t, as the expected return on assets ($EROA_t$) subject to some restrictions. We assume that $EROA_t$ can be constructed by adding a risk premium to the long-term interest rate. From Table 2.1 panel (a), we find that the historical risk premium over the 10-year bond yields is 3.27% for Canadian equities and 4.66% for the U.S. equities, expressed as annual effective rates. Using the assumed asset mix, for a valuation performed at time t, the expected return on assets is projected to be:

$$\begin{aligned} EROA_t &= 0.35 \times i_t^{120} + 0.20 \times (i_t^{120} + 3.27\%) + 0.45 \times (i_t^{120} + 4.66\%) \\ &= i_t^{120} + 0.03. \end{aligned} \tag{3.11}$$

We let the valuation rate y_t^V equal $EROA_t$, subject to a lower bound of 4% and an upper bound of 10%:

$$y_t^V = min(max(EROA_t, 4\%), 10\%). \tag{3.12}$$

The general salary increase assumption (s_t^*) is based on the actual salary increases experienced by the plan, but with less volatility. Specifically, we set s_t^* to be the average of actual salary increase rates (s_t) from inception to the valuation in question:

$$s_t^* = \frac{1}{t+1} \sum_{h=0}^{t} s_h. \tag{3.13}$$

3.3.4 Valuation Method

We use the entry-age normal cost method as defined in Aitken (1996). Under this method, the actuarial liability AL_t^{EAN} has three components:

$$AL_t^{EAN} = PVB_t + PVFB_t - PVFNC_t,\ t = 0, 3, 6, ... \tag{3.14}$$

PVB_t is the present value of total benefit payments to be made from the plan to members who have already retired by time t:

$$PVB_t = \sum_{x \geqslant r} \sum_{k \leqslant t/3} B_{x,t,k} \times \ddot{a}_{\overline{r+l-x|}y_t^V}. \tag{3.15}$$

$PVFB_t$ is the present value of total future benefits payable to members who are still active at time t:

$$PVFB_t = \sum_{x<r} \sum_{k \leqslant t/3} B^*_{x,t,k} \times \ddot{a}_{\overline{l|}y_t^V} \times \frac{1}{(1+y_t^V)^{r-x}} \tag{3.16}$$

where

$$B^*_{x,t,k} = 2\% \times (r-e) \times FAS^*_{x,t,k}. \tag{3.17}$$

and $FAS^*_{x,t,k}$ is the member's projected salary at retirement based on the actual salary ($S_{x,t,k}$) at time t and the salary increase assumption (s^*_t) applicable in the valuation at time t.

Finally, $PVFNC_t$ is the present value of future normal costs in respect of active members at time t:

$$PVFNC_t = \sum_{x<r} \sum_{k \leqslant t/3} c^{NC}_t \times Sal_{x,t,k} \times \ddot{a}_{\overline{r-x}|i^*_t}, \tag{3.18}$$

where c^{NC}_t is the normal cost rate established in the current valuation, and $i^*_t = \frac{1+s^*_t}{1+y^V_t} - 1$ is the valuation rate net of the salary increase assumption.

Under the EAN method, the normal cost is calculated as the level, long-term rate of pay required to finance the benefits of new entrants to the plan over their working lifetimes, so that their projected benefits are fully secured by equivalent assets by the time they retire (Aitken, 1996). Consequently, c^{NC}_t is the total present value of projected pension benefits for all new members entering the plan at time t, divided by the total present value of projected future salaries for the same new entrants. Mathematically,

$$c^{NC}_t = \frac{\sum_{x<r} B^*_{x,t,k} \times \ddot{a}_{\overline{l}|y^V_t} \times \frac{1}{(1+y^V_t)^{r-x}}}{\sum_{x<r} Sal_{x,t,k} \times \ddot{a}_{\overline{r-x}|i^*_t}}, \text{ for } k = t/3. \tag{3.19}$$

Starting salaries for new entrants at $t = 0$ are shown in Table A.1. Starting salaries for members who enter the plan in a subsequent valuation (i.e., $k \geqslant 1$) are the entry salaries shown in Table A.1 increased at a rate of 4% per 3 years (the time elapsed between each valuation).

3.3.5 Setting the Contribution Rate

We follow an approach very similar to the College Pension Plan, ignoring the Inflation Adjustment Account. To calculate the contribution rate, both the normal cost and the funded position of the plan at the valuation date are considered. To the extent actuarial assumptions are realized, the addition of new entrants to the plan should generate neither unfunded liabilities nor surplus. To determine the funded position, we compare plan liabilities to plan assets. The resulting net financial position may be either an actuarial surplus or an unfunded actuarial liability. Formally,

$$Surp_t = AV_t - AL^{EAN}_t. \tag{3.20}$$

A positive $Surp_t$ means the plan has surplus, while a negative value means the existence of an unfunded liability.

Newly emerging unfunded liabilities are amortized by special payments spread over 15 years from each valuation date. Total contributions in this case are equal to the total normal cost plus special payments.

When there is a gain since the last valuation, we apply the gain to reduce the previously established special payments proportionally. In the case that there is a surplus after removing all previously established special payments, we allocate up to 5% of the net liability to a buffer required under the BC pension regulations and refer to the remaining surplus, if any, as "usable surplus". We determine two possible contribution reduction amounts by amortizing the usable surplus over a 15-year period and over a 25-year period. We establish the minimum contribution rate as the normal cost less the 15-year amortization of surplus. We also establish the maximum contribution rate as the normal cost less the 25-year amortization of surplus. We then apply the following algorithm.

1. If the contribution rate determined in the last valuation is lower than the minimum contribution rate determined in the current valuation, then the rate should be increased to be equal to the minimum contribution rate, resulting in 15-year amortization of the usable surplus.

2. If the contribution rate determined in the last valuation is greater than the maximum contribution rate determined in the current valuation, then the contribution rate is reduced to the maximum level, resulting in much slower (25-year) amortization of the usable surplus.

3. If the contribution rate determined in the last valuation is between the minimum and maximum contribution rates determined in the current valuation, then the rate should remain unchanged. Effectively, some of the surplus which could have been spent over the next 15 years remains unspent and is applied as an additional rate stabilization reserve.

With the above contribution adequacy testing method, we get the minimum contributions (c_t) expressed as a percentage of payroll. Negative c_t means the fund pays back to plan members, which is not realistic. Therefore, the contribution rate is set to the greater of the projected value and 0. In this way, we adjust c_t carefully and avoid having fluctuating c_t between valuations. Detailed formulas for the amortization technique are provided in Appendix C.

3.4 Alternative Designs

Contribution stability is an important objective of the College Pension Plan. Additional stabilization reserves have been added onto surplus, and unfunded liabilities are amortized over a long period. On the asset side, the five-year smoothing method with corridors cushions the valuation results against dramatic swings in market value. We are interested in the effect that each of these stabilization techniques has on our results.

We refer to the stylized JSPP described in this chapter as JSPP1, which includes all sophisticated rate stabilization techniques. We also consider JSPP2, where the actuarial value of assets is replaced by market value. Finally, JSPP3 uses the market value of assets and applies 15-year amortization to both surplus and unfunded liabilities, without a rate stabilization reserve.

Table 3.5: Summary of average projected normal cost and contribution rates for the stylized JSPP

(a) average normal cost rates						
Year	2017	2020	2023	2026	2029	2032
c_t^{NC} (%)	23.01	17.38	15.03	13.65	12.84	12.27
Year	2035	2038	2041	2044	2047	2050
c_t^{NC} (%)	11.89	11.58	11.48	11.38	11.35	11.23
(b) average contribution rates c_t (%)						
Year	2017	2020	2023	2026	2029	2032
JSPP1	23.01	15.37	12.03	10.37	9.68	9.21
JSPP2	23.01	15.15	11.45	9.67	8.88	8.40
JSPP3	23.01	14.44	10.87	9.35	8.84	8.48
Year	2035	2038	2041	2044	2047	2050
JSPP1	8.76	8.40	8.15	7.91	7.67	7.23
JSPP2	7.96	7.83	7.63	7.58	7.55	7.37
JSPP3	8.08	8.00	7.83	7.82	7.85	7.59

Table 3.5 shows the average projected normal costs and contribution rates. Rates are relatively high in the first years, which is consistent with the pattern of simulated economic scenarios in the VAR model. As bond yields increase, the pension fund is projected to experience significant investment gains in the first 20 years leading to contributions that are lower than the normal cost. As rates rise, average normal costs shift downward. After economic variables return to their historical means and the membership profile stabilizes, both rates become stable. Note that the stabilized long term contribution rates are lower than the corresponding normal cost rates. This is due largely to the slow amortization of surplus, since unspent surplus generates additional gains for the plan at each valuation.

From JSPP1 to JSPP2, variation on the asset side increases. As we are more likely to experience gains in the first years, average contribution rates are reduced by a greater proportion. From JSPP2 to JSPP3, surplus is fully amortized without a rate stabilization reserve; therefore, contribution rates are reduced more quickly at first. However, in the long term, the stable contribution level is higher under JSPP3 than under JSPP2. This is because the rate stabilization reserve, which is built up from a portion of the surplus emerging in the early years, generates additional surplus whenever the rate of return on the pension fund is positive. The absence of these "gains" under JSPP3 means slightly higher contributions in the long run. Since our stylized plan starts from a position without any surplus, a portion of the early gains under JSPP1 and JSPP2 goes to build up the rate stabilization reserve, which benefits later cohorts. This represents a value transfer from early cohorts to later cohorts, which does not occur under JSPP3. Those who join JSPP3 early benefit from investment gains through the reductions on their contributions; those who join the plan 30 years from now are required to make more contributions, compared to members with same ages but under JSPP2.

Chapter 4

Simulation of Stylized DC Plan

4.1 The SFU Academic Pension Plan

The Simon Fraser University Pension Plan for Academic Staff was introduced in 1969 to provide retirement benefits for faculty members of the university. The current SFU plan is a non-contributory DC plan. Key features summarized below are from SFU Pension Plan for Academic Staff Summary (SFU, 2013) and SFU 2015 Annual Report Academic Pension Plan (SFU, 2015).

1. Contributions:
   The university contributes 10 percent of a member's basic salary, less $419.4 (representing the university's required contributions to the Canada Pension Plan back in 1986) to a fully vested individual account. The plan permits members to make voluntary contributions by regular payroll deduction up to the limits established in the Income Tax Act and Regulations. Currently only about 3% of members make voluntary contributions, and those who do contribute make an approximate 5% contribution.

2. Investment Choices:
   The funds in each member's account are invested under the direction of professional money managers monitored by the trustees of the SFU plan. According to SFU (2015b), approximately 81.6% of total account balances are invested in the SFU Balanced Pension Fund. Annual returns were 6.3% in 2015, 11.5% in 2014 and 18.8% in 2013 on the SFU Balanced Pension Fund, before taking into account the administration, consulting or investment management fees. In addition to the Balanced Pension Fund, a small number of other investment funds are offered to members, including money market funds, bond funds, as well as Canadian and foreign equity funds. Some of these are index funds while others are actively managed.

3. Retirement Dates:
   Normal retirement dates are the first day of September following a member's 65th birthday. Early retirement is allowed on the first day of any month after attained age 55.

4. Benefits on Retirement:
   Members can apply their accumulated funds to the purchase of an annuity, or move their account balance to another registered plan.

4.2 Stylized DC Plan

In line with the SFU plan, our stylized DC plan sets up an individual account for each plan member, and applies the balance, consisting of the accumulated contributions and investment earnings, to purchase a guaranteed annuity at the member's retirement date. However, instead of contributing a fixed percentage of salary, we assume contributions to the DC account are made at the same rates as to our stylized JSPP. That is, contribution rates can fluctuate from year to year and scenario to scenario.

4.2.1 Contributions

Annual contributions are made to the DC account as a percentage, c_t, of a member's annual salary, where c_t is as in the stylized JSPP. We also apply the same investment strategy as under the stylized JSPP with annual portfolio return r_t^P on $[t-1, t)$.

4.2.2 Pension Benefits

The DC plan member uses the accumulated account value to purchase a guaranteed annuity from an insurance company upon his retirement. To project annuity purchase rates applicable in future years, we refer to the work of the CIA Committee on Pension Plan Financial Reporting (PPFRC). Every quarter, the PPFRC issues guidance regarding the "appropriate discount rate for estimating the cost of purchasing a non-indexed group annuity" (CIA, 2016, p. 2). The discount rate is determined as the unadjusted average yield on Government of Canada marketable bonds with maturities over 10 years increased arithmetically by a spread that varies quarter to quarter.

Figure 4.1 (a) shows the historical benchmark bond yields (CANSIM series V39062) and the spreads recommended by the PPFRC (CIA, 2016). The "Annuity Conversion Rate" series in panel (b) is the sum of the two series from panel (a), representing the annuity purchase rate recommended by the PPFRC. The series labeled "10-year Bond" uses the same historical data as our VAR model. We see from panel (b) that the gap between the annuity purchase rate and the 10-year zero coupon yield is consistently around 1%.

Therefore, we estimate the annuity purchase rate applicable at time t, y_t^A, as:

$$y_t^A = i_t^{120} + 0.01 \tag{4.1}$$

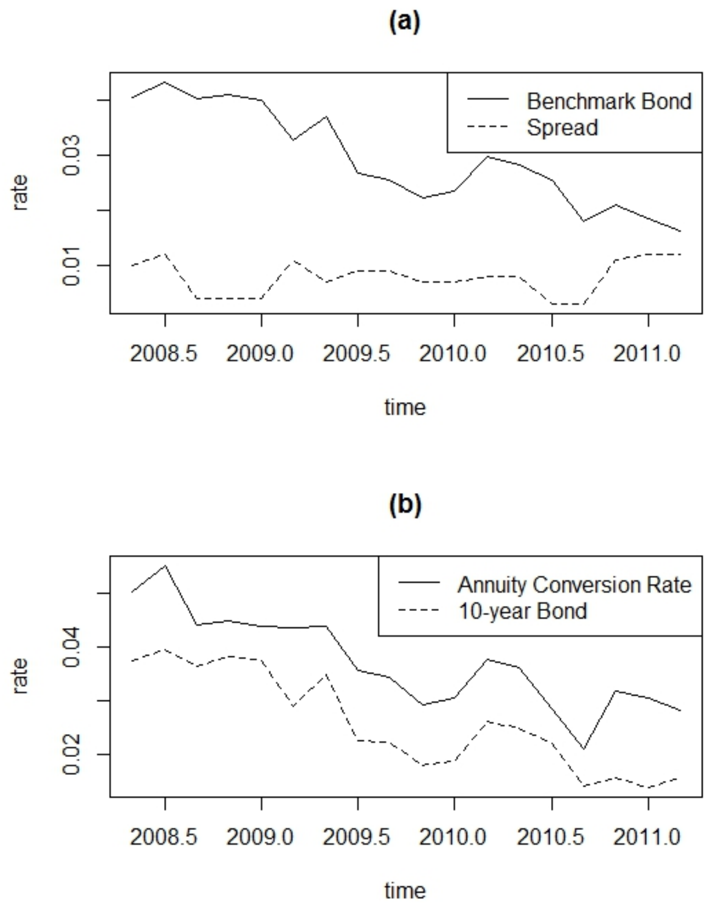
where i_t^{120} is the simulated 10-year bond yield from the VAR model, expressed as an annual effective rate.

4.3 Alternative Design

One disadvantage of DC plans in the current low interest rate environment is that the cost of annuities is very high. Generally the annuity purchase rate charged by an insurance company is lower compared to the valuation rates used in pension funds. There are several reasons for this: insurance companies are conservative investors, they need to build in margins for risk, expenses and profits. Also, unlike pension funds, insurance companies are limited in their abilities to transfer risks onto future generations of policyholders.

The stylized DC plan described above is named DC1. To help us investigate the impact of the difference between annuity purchase rates charged by insurance companies and the higher valuation rates used by pension funds, we add a design called DC2, under which the annuity at retirement is purchased at the corresponding valuation rate in JSPP1.

Figure 4.1: Relationship of annuity purchase rate and bond yields



Source: Statistics Canada. CANSIM series V39062. Table 176-0048: Bank of Canada, money market and other interest rates, daily; Canada; Government of Canada marketable bonds, over 10 years. `http://www.bankofcanada.ca/rates/interest-rates/lookup-bond-yields/`. Accessed October 9, 2016.

Chapter 5

Performance Evaluation

5.1 Comparison Criteria

We investigate the performance of the two pension plans for identical twins. One twin joins the stylized JSPP, makes varying contributions to the fund, and collects guaranteed benefits after retirement; the other twin follows the same contribution pattern, manages the money in his own account, and transfers the accumulated savings to purchase an annuity certain with the same period as his sibling's pension benefit. Both twins are assumed to be 30 years old at entry with annual salary of $70,000.

We use two metrics to compare outcomes. The first uses the pension ratio, which is the ratio of DC pension to JSPP pension. Unlike in Blake, Cairns and Dowd (2001), the benefits under our JSPP1 and DC1 are directly comparable because they have the same contribution patterns. Our simulations generate an empirical distribution of possible pension ratios. The values of the pension ratios range from 0.40 at the lower end to 6.24 at the upper end. To make a comparison, we apply value-at-risk, which is widely used in studying tail risks. We specify one or more percentiles from our distribution, and compare these values with a target pension ratio of 1. The i^{th} percentile is the VaR at the $(100-i)^{th}$ percent confidence level. If this percentile is greater than or equal to 1, we would conclude that the DC plan successfully replicates the benefits available from JSPP; and vice versa. A DC plan will never replicate a JSPP with 100% confidence level, and our concern is the amount of risk an individual carries to achieve a certain level of pension benefit.

The second metric is the expected discounted utility of pension benefits. Expected discounted utility is used to perform welfare comparison among various plans. Pension plans that lead to higher consumption levels are ranked higher in utility terms. In using utility theory, an important consideration is the choice of the utility function. We use power utility which has a constant relative risk aversion:

$$u(Con_t) = \frac{Con_t^{1-\gamma}}{1-\gamma}, \tag{5.1}$$

where Con_t is the consumption at time t. Prior to retirement, the twins have the same salary and the same contribution patterns, so their consumption is the same. Therefore, we only need to consider consumption during the retirement years, which is the annual pension payment B. The parameter γ represents the constant relative risk aversion level. We choose a relatively conservative $\gamma = 5$ which implies that "workers are ready to pay as much as 2.4% of their wealth to eliminate a fifty–fifty risk to gain or loose 10% of their wealth" (Gollier, 2008).

The expected discounted utility of benefits is:

$$E_{r-e}(U) = E\Big[\sum_{t=0}^{l-1} e^{-\beta t} u(\hat{B})\Big], \tag{5.2}$$

where β is the individual's time preference rate chosen as 0.04 following Cui et al. (2011), and $\hat{B}$ is the benefit received at time t, adjusted for the effect of inflation during the member's working years. Under the stylized JSPP, a high inflation scenario during a member's working years will result in higher final salary, translating to a higher benefit and higher consumption. However, such a scenario does not necessarily provide better welfare, because the purchasing power of this benefit is lower than the purchasing power of the same nominal benefit under scenarios with lower inflation. To correct for this, we adjust benefits by stripping out the impact of stochastic inflation between time 0 and retirement, and apply the utility function to the adjusted benefit. The adjustment is different from scenario to scenario, but is applied consistently between the stylized JSPP and the stylized DC plan.

Specifically, the unadjusted annual benefit under the stylized JSPP, B^{JSPP}, is calculated as:

$$B^{JSPP} = 2\% \times (r - e) \times FAS, \tag{5.3}$$

where FAS is based on the actual salary increases given in equation (3.1). The corresponding adjusted annual benefit $\hat{B}^{JSPP}$ that enters the power utility has the following form:

$$\hat{B}^{JSPP} = 2\% \times (r - e) \times F\hat{A}S, \tag{5.4}$$

where $F\hat{A}S$ is calculated under the assumption that salary increase rate is only 0.75% per year. The adjustment factor $Adj = \hat{B}^{JSPP}/B^{JSPP}$ is applied to the annual DC benefit B^{DC} in the same scenario. As a result, the adjusted annual DC plan benefit, $\hat{B}^{DC}$, that enters the power utility has the following form:

$$\hat{B}^{DC} = Adj \times B^{DC}. \tag{5.5}$$

5.2 Numerical Results: Benchmark Comparison

We first look at the benchmark case (DC1 versus JSPP1). Value-at-risk statistics are in the first row of Table 5.2 panel (a). On average, the DC twin receives 1.59 times the retirement pension of his JSPP twin. If we want a reliable indicator of how risky the DC plan can be, we can look at the 5% quantile which is 0.8564. It indicates a 5% chance that the pension ratio will be less than 86%. However, as the required confidence level decreases, the DC plan becomes more attractive. For example, if we take the 75% confidence level, then the DC plan outperforms JSPP. The confidence level at which the DC twin's pension is the same as the JSPP twin's (i.e., a VaR of 1) is 88.07%. In conclusion, whether or not the DC plan is more competitive than the JSPP will depend on the choice of VaR confidence level.

The expected discounted utility summarized in Table (5.2) panel (b) supports that, with the same contribution pattern, the DC plan can be a better choice from a welfare perspective, based on the assumed risk tolerance of the twins. Note that the choice of time preference rate, β, does not affect the ordering of the pension plans because the benefits are fixed after retirement, so the terms relating to β can be factored out of the expected value:

$$\begin{aligned} E_{r-e}(U) &= E\Big[\sum_{t=0}^{l-1} e^{-\beta t} u(\hat{B})\Big] \\ &= E\Big[u(\hat{B}) \cdot \ddot{a}_{\overline{l|}\beta}\Big] \\ &= \ddot{a}_{\overline{l|}\beta} \cdot E\Big[u(\hat{B})\Big]. \end{aligned} \tag{5.6}$$

As a result, changing β changes the expected discounted utility of both options by the same proportion.

5.3 Numerical Results: Alternative Designs

To recap, we display the alternative JSPP and DC designs in Table 5.1. From DC1 to DC2, the annuity purchase capacity is improved with higher valuation rates. From JSPP1 to JSPP2, asset values experience more volatility. From JSPP2 to JSPP3, surplus in the valuation is amortized sooner and more completely.

Table 5.1: Comparison of alternative DC and JSPP designs

	DC1	DC2	JSPP1	JSPP2	JSPP3
Membership	Single Member		Full Plan[1]		
Asset Value[2]	MV	MV	AV	MV	MV
Valuation Rate	y_t^A	y_t^V	y_t^V	y_t^V	y_t^V
Amortization of surplus & deficits	None	None	15 years for deficits; $15-25$ years for spendable surplus with rate stabilization reserve	15 years for deficits; $15-25$ years for spendable surplus with rate stabilization reserve	15 years for both deficit & spendable surplus

[1] Full Plan = shifting demographic profile.
[2] MV = market value; AV = smoothed actuarial value with 92%/108% corridors.

Value-at-risk statistics are shown in Table 5.2 panel (a) and used to plot the empirical cumulative distribution function of the pension ratio for different designs on Figure 5.1. Comparison results of expected utility for DC1, DC2 and JSPP1 are summarized in Table 5.2 panel (b). We summarize our findings below.

- Each of the empirical distributions shows some degree of tail risk whichever contribution design it follows. Pension ratios drop and tail risk increases when the contribution pattern under the JSPP (and correspondingly the DC plan) is more volatile. For example, if the JSPP twin joins a plan that does not smooth asset value or cushions surplus (i.e., JSPP3), then the DC twin has only an 84.94% confidence level that his retirement benefit will be as good or better, while this confidence level increases to 88.07% when comparing twins who participate in JSPP1 and DC1.

- Results also confirm that some advantages come from a higher valuation rate allowed in pension funds. Under each JSPP design, a higher annuity purchase rate improves the performance of stylized DC plans significantly. For example, we have a 95.28% confidence level that DC2 will provide the same pension benefit as in JSPP1, while the corresponding confidence level is 88.07% under DC1.

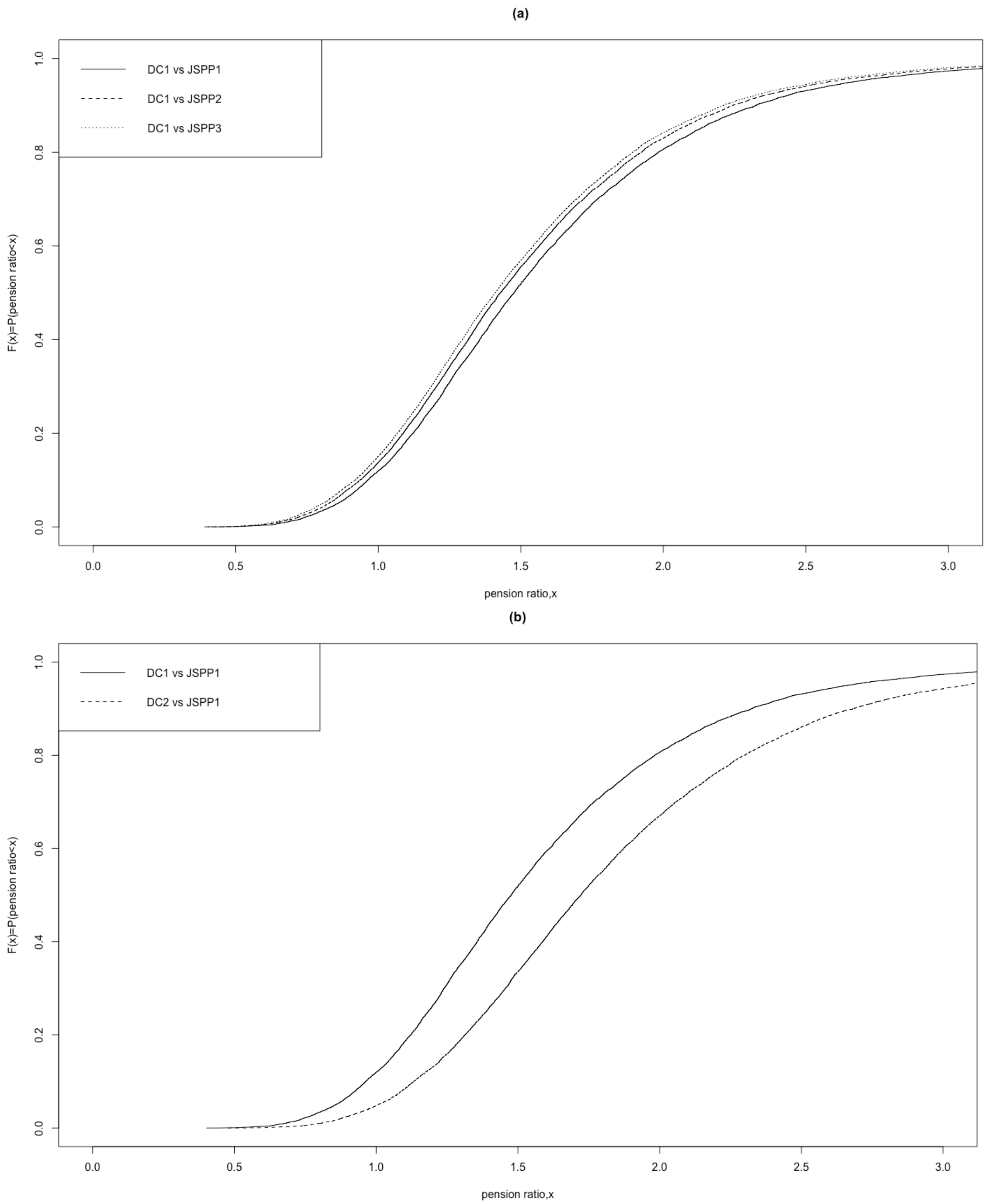
- Figure 5.1 shows the empirical cumulative distribution functions of the pension ratio for different pairs of pension plan designs. A curve that lies further to the right means that the DC design is more likely to deliver higher pensions at retirement than the corresponding JSPP design. From Figure 5.1 (a), the DC plan following the contribution patterns of JSPP3 shows less advantages than the DC plan following the contribution patterns of JSPP1. In this sense, the rate stabilization structure has secured the stylized JSPP both on the asset and liability sides. From Figure 5.1 (b), available benefits increase if the DC twin is able to purchase his retirement annuity at a better price.

- We can only compare expected discounted utility under the same contribution pattern. Results in Table 5.2 panel (b) prove that a cheaper annuity at retirement leads to welfare gains.

Table 5.2: Pension ratios and expected utility for different DC and JSPP designs

(a) value-at-risk statistics							
	Mean	SD	VaR				Critical Value 1
			50%	75%	90%	95%	VaR
DC1 vs JSPP1	1.5857	0.5887	1.4762	1.1835	0.9637	0.8564	88.07%
DC1 vs JSPP2	1.5368	0.5709	1.4284	1.1485	0.9361	0.8260	86.23%
DC1 vs JSPP3	1.5124	0.5630	1.4104	1.1285	0.9181	0.8050	84.94%
DC2 vs JSPP1	1.8441	0.6669	1.7220	1.3880	1.1326	1.0070	95.28%
DC2 vs JSPP2	1.7873	0.6467	1.6670	1.3448	1.0999	0.9711	94.11%
DC2 vs JSPP3	1.7589	0.6380	1.6461	1.3240	1.0756	0.9480	93.25%
(b) expected discounted utility (salary scaled to 1/100,000)							
	$E_{r-e}(U)$						
DC1	-18.00						
DC2	-9.43						
JSPP1	-35.22						

Figure 5.1: Cumulative distribution functions for different DC and JSPP designs (x_0)



5.4 Numerical Results: Alternative Economic Condition

To answer the question of whether today is the right time to switch from a DC plan to a JSPP, we investigate the case that the twins join the stylized plans when economic variables are above their long-term means. Specifically, simulations of economic scenarios start from a state as far from the long term mean as we are today, but in the opposite direction. More precisely, we replace the time 0 state variable

$$x_0 = \mu - (\mu - x_0)$$

with

$$x_0{}' = \mu + (\mu - x_0) = 2\mu - x_0.$$

The resulting average normal costs and contribution rates are listed in Table 5.3 panels (a) and (b). Value-at-risk statistics and expected discounted utility can be found in panels (c) and (d). Figure 5.2 shows the empirical cumulative distributions of the pension ratios. Our observations are as follows.

- Normal cost rates show a gradual upward shift in the first 20-25 years, because interest rates start at a high value and tend to decrease. In the long term, the average normal cost rates reach equilibrium at a similar level as in the original simulations. As yield rates fall, the stylized JSPP fund experiences significant losses at first, which results in increases in the contribution rates. JSPP1 is still the least volatile design. Since the difference between JSPP2 and JSPP3 is only related to surplus scenarios, average contribution rates look identical in the first few valuations.

- If the twins set up their pension plans in a high-interest-rate environment, the DC plan performs very well. The benchmark case shows an over 95% confidence that the pension ratio will be greater than 1. If the DC twin finds a better annuity purchase deal as under DC2, this level is as high as 98%. This can be explained by the fact that, with a upward shift in contribution rates, the DC twin maintains a smaller account value in the first few years, when investment markets are more volatile. As economic variables approach their historical means, the DC twin gradually increases his contributions, and the accumulated account balance secures him a more stable retirement benefit. There is only a slight change moving from JSPP2 to JSPP3, which is due to the similar contribution rates under the two designs.

- The shapes and positions of the cumulative distribution function curves are similar to those for the simulations starting below the long-term means of economic variables. Figure 5.2 (a) indicates that the advantages of JSPP designs come from their rate stabilization techniques. Figure 5.2 (b) confirms that a cheaper annuity at retirement improves the DC pension ratio since the cumulative distribution function curve moves to the right.

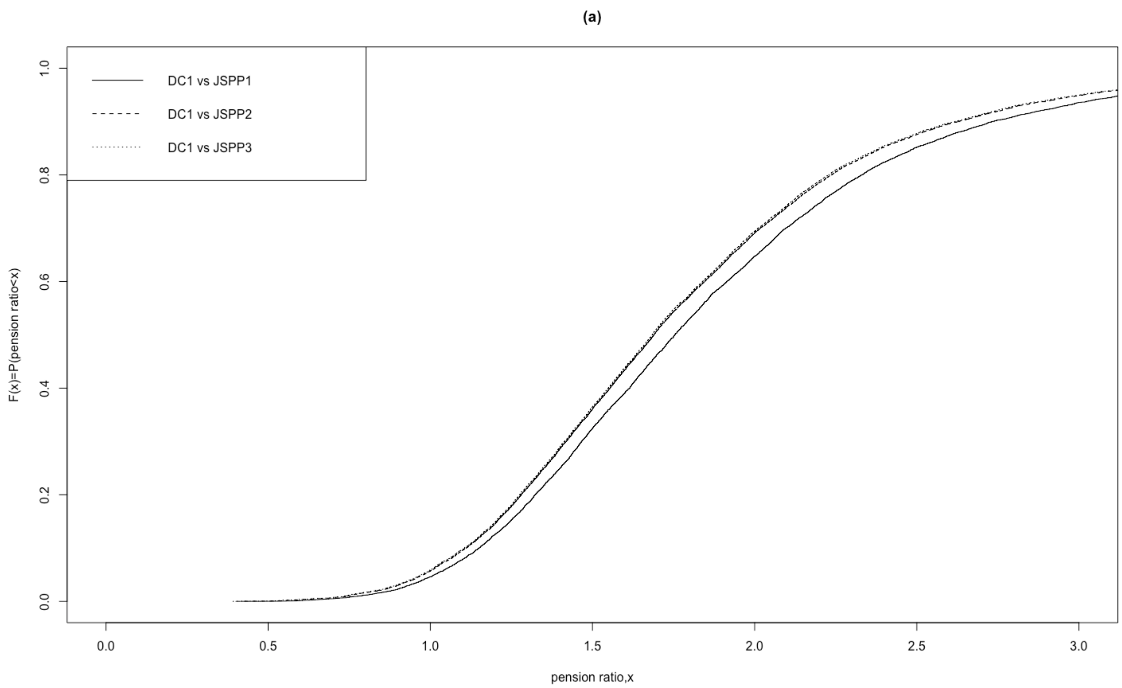
- The expected discounted utility ranks the DC1, DC2 and JSPP1 designs in the same order as before. From a welfare perspective, the DC plan is more appealing, and a higher annuity purchase rate will transform the accumulated account value into a better lifetime benefit.

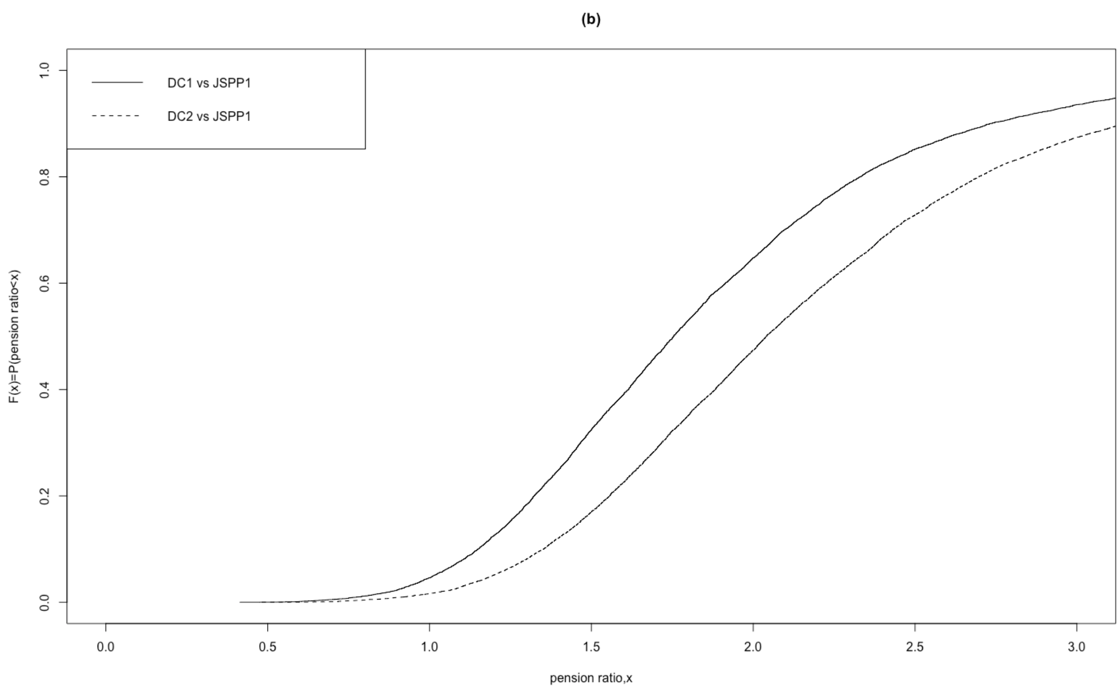
Table 5.3: Projected rates, pension ratios and expected utility for different DC and JSPP designs

(a) average normal cost rates						
Year	2017	2020	2023	2026	2029	2032
c_t^{NC} (%)	5.55	7.95	8.42	9.10	9.70	10.21
Year	2035	2038	2041	2044	2047	2050
c_t^{NC} (%)	10.43	10.66	10.81	10.91	10.96	10.96
(b) average contribution rates c_t (%)						
Year	2017	2020	2023	2026	2029	2032
JSPP1	5.55	11.82	14.32	14.95	14.81	13.81
JSPP2	5.55	11.76	12.91	13.42	13.59	13.07
JSPP3	5.55	11.76	12.91	13.42	13.59	13.12
Year	2035	2038	2041	2044	2047	2050
JSPP1	12.66	11.81	11.17	11.06	10.64	10.23
JSPP2	12.56	12.18	11.73	11.65	11.51	11.16
JSPP3	12.59	12.22	11.79	11.78	11.64	11.33

(c) value-at-risk statistics							
	Mean	SD	VaR				Critical Value 1
			50%	75%	90%	95%	VaR
DC1 vs JSPP1	1.8753	0.6914	1.7542	1.4004	1.1505	1.0123	95.39%
DC1 vs JSPP2	1.8033	0.6575	1.6893	1.3517	1.1104	0.9789	94.29%
DC1 vs JSPP3	1.7964	0.6548	1.6845	1.3460	1.1057	0.9724	94.11%
DC2 vs JSPP1	2.1784	0.7837	2.0409	1.6394	1.3516	1.1947	98.38%
DC2 vs JSPP2	2.0948	0.7452	1.9664	1.5823	1.3028	1.1535	97.87%
DC2 vs JSPP3	2.0869	0.7422	1.9607	1.5778	1.2976	1.1464	97.77%
(d) expected discounted utility (salary scaled to 1/100,000)							
	$E_{r-e}(U)$						
DC1	-9.56						
DC2	-5.02						
JSPP1	-35.22						

Figure 5.2: Cumulative distribution functions for different DC and JSPP designs ($x_0{}'$)





Chapter 6

Alternative Performance Criterion

The comparison metrics in Chapter 5 suggest that the stylized DC plan is quite attractive. The pension ratios show that the DC twin is expected to receive a 50% higher pension than his JSPP sibling on average, while only experiencing a moderate degree of downside risk. Even for a risk averse individual, the stylized DC plan provides higher expected discounted utility. It seems that the stylized JSPP has not met the goal of providing better retirement benefits through efficient pooling of risks, at least for the cohort of new employees we consider here. This is partly due to the simplifying assumptions we make in setting up the stylized JSPP. Without considering the inflation adjustment account, we ignore an important feature of the College Pension Plan which greatly affects an individual's risk and reward. Another reason the DC plan outperforms the JSPP is that we assume the DC twin follows the same contribution pattern as his JSPP sibling.

The time varying contribution rates determined in the triennial valuations of the JSPP are responsive to economic conditions: rates drop in response to gains and increase in response to losses, ensuring that the plan can deliver on its benefit promise over time. The sophisticated stabilization techniques smoothe out the short term fluctuations in the contribution rates. By following the same contribution pattern, the DC twin effectively optimizes his contribution pattern, while being free of the potential burden of subsidizing other cohorts. This produces superior outcomes. However, it is not realistic to assume that an individual DC plan member would react the same way to economic dynamics as the JSPP does. In fact, most DC plans have flat contribution rates.

Using a flat contribution rate on the DC side makes it difficult to compare the two plans. The pension ratio is no longer applicable because benefits flowing from different contributions under the DC plan and the JSPP are not comparable. Many studies use expected discounted utility, but it has several problems: it is very difficult to choose appropriate risk parameters, and individuals tend to have varying risk preferences at different life stages. Instead of using the utility framework which determines the subjective value attributed to each plan by an individual, an alternative approach could be market-based measurement of the net value that each plan provides. Specifically, the methodology used in Hoevenaars and Ponds (2008) and Lekniute at al. (2014) can be applied to estimate the value that market participants may pay for the net benefit stream, taking into account risks. This requires the estimation of the market price of risk, which can be achieved by extending the VAR model with an affine term structure model of interest rates (Cochrane and Piazzesi, 2005). The resulting model, which we constrain to being arbitrage free, produces a stochastic discount factor (or pricing kernel), which can be used to discount both future contributions and benefits. In the rest of this chapter we outline how this could be implemented based on Cochrane and Piazzesi (2005) and Hoevenaars (2008).

As suggested in Cochrane and Piazzesi (2005), the market prices of risk can be generated in an affine model. In line with their VAR model which contains an intercept term, we rewrite equation (2.1) as:

$$x_{t+1} - \mu = \Phi(x_t - \mu) + P\epsilon_{t+1}, \tag{6.1}$$

and therefore,

$$x_{t+1} = (I - \Phi)\mu + \Phi x_t + P\epsilon_{t+1}, \tag{6.2}$$

and we use $\nu = (I - \Phi)\mu$. The state variables included in x_t would need to be slightly different than described in Chapter 2. Stock return variables should be returns in excess of the short term interest rate and represent only the price appreciation. The corresponding dividend yields, considered as non-tradable assets on which the risk premium is zero, should appear as separate state variables rather than being part of the total equity return.

The pricing kernel has the following form:

$$M_{t+1} = exp(- \delta_0 - \delta_1 x_t - \frac{1}{2}\lambda_t^T PP^T \lambda_t - \lambda_t^T P\epsilon_{t+1}), \tag{6.3}$$

where M_{t+1} is the one-period stochastic discount factor, and $\delta_0 + \delta_1 x_t$ is the short rate which is affine in the state variables of the VAR. The innovation term ϵ_{t+1} is the same as in equation (6.2). To keep consistency between the VAR model and the pricing kernel, we let $\tilde{y}_t^1 = \delta_0 + \delta_1 x_t$. The first part of the stochastic discount factor, $exp(- \delta_0 - \delta_1 x_t)$, represents the risk-free discount factor. The other component, $exp(- \frac{1}{2}\lambda_t^T PP^T \lambda_t - \lambda_t^T P\epsilon_{t+1})$, relates shocks in the state variables to the pricing kernel. We use λ_t to represent the market price of risk and assume it has the following form:

$$\lambda_t = \lambda_0 + \Lambda_1 x_t. \tag{6.4}$$

The vector λ_0 accounts for the constant part of the risk premium, and the matrix Λ_1 accounts for time-variation. Since the market price of risk is the excess expected return per unit of covariance, λ_t is strictly positive. If the state of the economy is such that the market price of risk is high, the stochastic discount factor in (6.3) assumes a low value, all other things being equal.

The following section is in line with the description in Hoevenaars (2008). Asset pricing theory states that the price of an asset (P_t) is its expected discounted payoff:

$$P_t = E_t(M_{t+1}X_{t+1}), \tag{6.5}$$

where X_{t+1} is the asset payoff. The price $P_t^{(n)}$ of an n-period nominal bond at time t has the form of:

$$P_t^{(n)} = E_t(M_{t+1}P_{t+1}^{(n-1)}). \tag{6.6}$$

At the same time, the bond price can be expressed as an exponential affine function of the state variables in the VAR model. More precisely, bond prices are given by

$$P_t^{(n)} = exp(A_n + B_n^T x_t), \tag{6.7}$$

and therefore, log bond prices $p_t^{(n)}$ becomes a linear function of the state variables:

$$p_t^{(n)} = A_n + B_n^T x_t. \tag{6.8}$$

The scalar A_n and the vector B_n follow the difference equations:

$$\begin{aligned} A_{n+1} &= A_n + B_n^T(\nu - PP^T\lambda_0) + \frac{1}{2}B_n^T PP^T B_n - \delta_0 \\ B_{n+1}^T &= B_n^T(\Phi - PP^T\Lambda_1) - \delta_1 \end{aligned} \tag{6.9}$$

with $A_0 = B_0 = 0$ as $p_t^{(0)} = 0$. These difference equations can be derived by induction using equation (6.8); see Appendix D for details. The equations above show that the constant part of the risk premium λ_0 influences A_n, and the time-varying component Λ_1 influences B_n.

Since bond yields are related to bond prices, the log bond yields can be written as $p_t^{(n)} = -ny_t^{(n)}$, and therefore,

$$y_t^{(n)} = -\frac{A_n}{n} - \frac{B_n^T}{n}x_t. \tag{6.10}$$

Ang and Piazzesi (2003) suggest a two-step estimation process. The parameters in the VAR model (ν, Φ and P) are estimated through ordinary least squares. Next, we estimate the risk parameters (i.e., λ_0 and Λ_1) conditional on the VAR parameters. Values of A_n and B_n are obtained recursively. In setting up the risk parameters, we assume that the risk premium is zero for inflation and dividend yields. Furthermore, the risk premia related to stock returns are assumed to be the same as under the VAR model. The remaining elements of λ_0 and Λ_1 are estimated by minimizing the difference between the estimated yields from equation (6.10) and historical yields at selected maturities. This modelling framework results in consistency between the return dynamics of the VAR model and the term structure of interest rates.

Once λ_0 and Λ_1 are estimated, we can simulate the state variable x_t as well as the stochastic discount factor M_t. Bond returns are calculated by equation (2.14), but the projected log price $p_t^{(n)}$ is based on equation (6.10). The new performance criterion is the expected stochastic present value of net benefits ($ESPV$), which is the market value at time 0 of benefits to be received from the plan less contributions to be paid to the plan, adjusted for risk. More precisely, $ESPV$ is the average of the stochastic present values determined under each scenario:

$$SPV = CF_0 + \sum_{t=1}^{r+l-e-1} \Big[\prod_{h=12(t-1)+1}^{12t} M_h \Big] \times CF_t \tag{6.11}$$

where M_h corresponds to the one-month stochastic discount factor applicable to month h with $h = 1$ corresponding to January 2017 and t is measured in years. A positive CF_t means a cash inflow (the benefit payment at time t), and a negative CF_t means a cash outflow (contribution made at time t). Under our stylized JSPP and stylized DC plan, there are only cash outflows before retirement and only cash inflows afterwards. A positive $ESPV$ means that an individual pays less than the market value of the benefit stream he could receive, and a negative $ESPV$ means that the pension plan requires an individual to pay more than the market value of what he will receive.

The market-based evaluation approach has the advantage of comparing various pension designs flexibly. There is no requirement for matching contribution streams so we can directly compare DC1, DC2, JSPP1, JSPP2 and JSPP3. Unlike expected discounted utilities which produce only a ranking, the numeric value of the $ESPV$ has a specific meaning. Consequently, the difference between the $ESPV$s of different pension designs can be interpreted as the increase (loss) of value to the new member (on a market-consistent basis) when moving from one design to the other. This means we could not only quantify the total difference in value between our stylized plans, but could also attribute that difference to various sources, such as the difference in annuity purchase prices under pension plans versus insurance companies, fixed versus variable contributions, individual versus collective plans, as well as the various smoothing mechanisms.

Chapter 7

Conclusion

In this study, we compare the performance of two stylized pension plans: a DC plan based on the Simon Fraser University pension plan for faculty members and a jointly-sponsored pension plan replicating some features of the B.C. College Pension Plan. A VAR(1) model is used to generate economic scenarios. We investigate pension ratios and expected discounted utilities for a representative cohort. We also conduct analysis on alternative plan designs and different economic conditions to understand the two plans more comprehensively. We find that the DC outcomes are volatile, but risk is mostly on the upside when contribution rates mimic those under the JSPP. In terms of expected discounted utilities, the DC plan wins even for a conservative member. In addition, the impact of smoothing mechanisms under the JSPP is quite small. Finally, the DC plan would do even better if insurance companies were subject to the same rates as pension plans and if simulations were started in "opposite" economic conditions.

In designing our stylized DC plan and our stylized JSPP, we make some simplifying assumptions. For example, expenses are completely ignored. In reality, the total cost of investment management and pension administration for large public funds tends to be much less than what DC plan members commonly pay. Another conservative assumption on the stylized JSPP is that investment returns are based on an index, while actual public sector pension plans offer more efficient and less volatile asset allocation strategies. In addition, we focus on the guaranteed basic pension benefit while conditional inflation protection is an important characteristic of JSPPs in general, and the College Pension Plan in particular.

Potential future work includes more sophisticated asset models, simulation of the Inflation Adjustment Account, as well as the implementation of the market-value based performance criterion described in Chapter 6.

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Appendix A

Membership of Stylized JSPP

The following three tables display membership information that we use to project the stylized JSPP.

Table A.1: New entrant data for the stylized JSPP

Male			Female		
Number	Age	Average Salary ($)	Number	Age	Average Salary ($)
16	24	57,641	39	24	54,422
115	27	65,486	219	27	62,449
133	30	69,966	173	30	67,180
133	33	69,966	174	33	67,180
135	36	72,767	185	36	70,982
135	39	72,767	185	39	70,982
271	42	75,537	334	42	72,732
110	45	76,052	139	45	70,083
110	48	76,052	139	48	70,083
86	51	79,805	125	51	76,484
86	54	79,805	126	54	76,585
124	57	80,279	145	57	73,601
103	60	80,641	84	60	74,013

Table A.2: Active member data for the stylized JSPP

	Male			Female		
Age	Number	Average Salary ($)	Service	Number	Average Salary ($)	Service
24	7	63,059	0.1	11	36,296	0.5
27	50	59,443	0.7	113	60,167	0.9
30	109	68,065	1.5	181	66,446	1.9
33	109	68,065	1.5	181	66,446	1.9
36	205	72,709	2.9	317	72,166	3.2
39	205	72,709	2.9	317	72,166	3.2
42	600	78,185	4.4	843	76,446	4.6
45	366	81,332	6.3	497	78,852	6.5
48	366	81,332	6.3	498	78,852	6.5
51	465	84,675	9.5	533	81,122	8.4
54	464	84,675	9.5	533	81,122	8.4
57	886	86,612	11	1038	83,193	10.8
60	1069	87,697	12.3	985	82,709	11.9

Table A.3: Pensioners data for the stylized JSPP

	Annual Pensions ($)	
Age	Male	Female
60	2,473,000	3,386,000
62	12,161,000	15,030,000
67	21,305,000	23,214,000
72	21,629,000	13,405,000
82	4,223,000	1,533,000
87	0	647,000

Appendix B

Asset Value Smoothing Method

We apply a five-year smoothing technique to the assets of the JSPP. To obtain the unconstrained smoothed value at time t, we first determine the actual portfolio return during year t on the basis of market values. We then determine an assumed return for the year, which is the valuation interest rate assumption from the previous valuation. The difference between the actual and assumed returns is then spread over a five-year period, recognizing one-fifth of it in each of the current and four succeeding years.

The smoothed value of assets is then restricted to a range of 92% to 108% of the market value of assets. This means that in periods of significant market decline (growth), the smoothed value does not become too large (low) relative to the market value; effectively the constraint accelerates recognition of very poor (strong) market returns and allows the contribution rate to more appropriately reflect the actual returns earned by the plan.

In this section we use the following notation:

MV_t : market value of JSPP assets at time t,

UAV_t : unconstrained actuarial (smoothed) value at time t,

AV_t : actuarial value at time t, after applying 92%/108% market value corridor,

G_t : investment gain (loss) arising at time t, in respect of $[t-1, t)$,

$UG_{t,0}$: unrecognized gain (loss) remaining at time t from gain (loss) first arising at t,

$UG_{t,1}$: unrecognized gain (loss) remaining at time t from gain (loss) first arising at $t-1$,

$UG_{t,2}$: unrecognized gain (loss) remaining at time t from gain (loss) first arising at $t-2$,

$UG_{t,3}$: unrecognized gain (loss) remaining at time t from gain (loss) first arising at $t-3$.

We also define:

$UG^*_{t,m}$: the unrecognized portion at time t of the gain (loss) first arising at time $t-m$, prior to the application of the market value corridor,

TUL_t : total unrealized gain (loss) at time t, prior to the application of the market value corridor,

DUL_t : desired unrealized gain (loss) at time t, after the application of the market value corridor.

When $t = 0$,

$$UG_{0,0} = UG_{0,1} = UG_{0,2} = UG_{0,3} = 0. \tag{B.1}$$

When $t = 1, 2, ...$

$$G_t = [MV_{t-1} + C_{t-1} - B_{t-1}](r_t^P - y_t^V) \tag{B.2}$$

$$UG^*_{t,3} = \frac{1}{2} UG_{t-1,2} \tag{B.3}$$

$$UG^*_{t,2} = \frac{2}{3} UG_{t-1,1} \tag{B.4}$$

$$UG^*_{t,1} = \frac{3}{4} UG_{t-1,0} \tag{B.5}$$

$$UG^*_{t,0} = \frac{4}{5} G_t \tag{B.6}$$

$$MV_t = [MV_{t-1} + C_{t-1} - B_{t-1}](1 + r_t^P) \tag{B.7}$$

$$UAV_t = MV_t - \sum_{i=0}^{3} UG^*_{t,i} \tag{B.8}$$

$$AV_t = min(max(UAV_t, 0.92MV_t), 1.08MV_t) \tag{B.9}$$

$$TUL_t = UAV_t - MV_t \tag{B.10}$$

$$DUL_t = AV_t - MV_t \tag{B.11}$$

$$UG_{t,m} = UG^*_{t,m} \times \frac{DUL_t}{TUL_t}, m = 0, 1, 2, 3. \tag{B.12}$$

Appendix C

Amortization Method

As discussed in section (3.3.5), any unfunded liability is amortized over the next 5 valuations (15 years). In the case of a surplus, only the surplus in excess of 5% of the net liability (the "usable surplus") is amortized. We determine new minimum and maximum contribution levels at each valuation. An increase of contribution rate is required if the contribution rate established in the previous valuation is smaller than the new minimum contribution rate. A reduction of contribution rate is approved if the contribution rate established in the previous valuation is greater than the maximum contribution rate. The contribution rate is unchanged otherwise.

Notations that are used in this section:

AV_h : actuarial value of assets established in valuation h (at time $3h$),

AL_h^{EAN} : actuarial liability established in valuation h,

$TSal_h$: total annual salaries in valuation h, equal to $\sum_x \sum_{k \leqslant h} Sal_{x,3h,k}$,

$Surp_h$: surplus (unfunded liability) from pure assets in valuation h,

$SP_{h,m}$: annual special payment first established in valuation $h-m$ to amortize an unfunded liability, and still applicable in valuation $h, (m = 0, 1, 2, 3, 4)$,

$Surp_h^*$: adjusted surplus, with present value of special payments being included as an additional asset,

spr_h^* : special payment rate from $Surp_h^*$ in valuation h,

spr_h : special payment rate from $Surp_h$ in valuation h,

c_h^{min} : the contribution rate arising after amortizing any usable surplus surplus over a 15-year period,

c_h^{max} : the contribution rate arising after amortizing any usable surplus surplus over a 25-year period,

c_h^* : contribution rate before adding the non-negative constraint in valuation h,

c_h : final contribution rate established in valuation h.

When $h = 0$,

$$Surp_0 = Surp_0^* = 0, \tag{C.1}$$

$$SP_{0,0} = SP_{0,1} = SP_{0,2} = SP_{0,3} = SP_{0,4}, \tag{C.2}$$

$$c_h = c_0^{NC}. \tag{C.3}$$

When $h = 1, 2, ...$

$$Surp_h = AV_h - AL_h^{EAN}. \tag{C.4}$$

If $Surp_h > 0$, the actuarial value of plan assets is enough to finance future liabilities without the need for any special payments in excess of the normal cost. We remove any previously established special payments:

$$SP_{h,m} = 0, m = 0, 1, 2, 3, 4, \tag{C.5}$$

$$spr_h = 0. \tag{C.6}$$

We establish the minimum and maximum contribution rates as the normal cost rate less the amortization of the usable surplus over a 15-year period and a 25-year period, respectively:

$$c_h^{min} = c_h^{NC} - \frac{max(Surp_h - 5\%AL_h^{EAN}, 0)}{\ddot{a}_{\overline{15|}} \times TSal_h}, \tag{C.7}$$

and

$$c_h^{max} = c_h^{NC} - \frac{max(Surp_h - 5\%AL_h^{EAN}, 0)}{\ddot{a}_{\overline{25|}} \times TSal_h}. \tag{C.8}$$

The contribution rate established in the previous valuation, c_{h-1}, is then compared to the minimum and maximum rates determined above:

$$c_h^* = \begin{cases} c_h^{min}, \text{ if } c_{h-1} < c_h^{min}, & \text{(C.9)} \\ c_h^{max}, \text{ if } c_{h-1} > c_h^{max}, & \text{(C.10)} \\ c_{h-1}, \text{ if } c_h^{min} \leqslant c_{h-1} \leqslant c_h^{max}. & \text{(C.11)} \end{cases}$$

Finally, we constrain the contribution rate to non-negative values:

$$c_h = max(c_h^*, 0). \tag{C.12}$$

If $Surp_h < 0$, the actuarial value of plan assets is not enough to finance future liabilities. We look at the adjusted surplus which takes into account the present value of special payments established in previous valuations, if any, which are still outstanding:

$$Surp_h^* = Surp_h + \sum_{m=1}^{4} SP_{h,m} \cdot \ddot{a}_{\overline{15-3m|}y_h^V}. \tag{C.13}$$

- If $Surp_h^* < 0$, the actuarial value of plan assets plus future special payments are still not enough to finance future liabilities, thus we need additional special payments. We let the new special payment be

$$SP_{h,0} = -\frac{Surp_h^*}{\ddot{a}_{\overline{15}|y_h^V}}, \tag{C.14}$$

and continue the special payments established in prior valuations as original scheduled:

$$SP_{h,m} = SP_{h-1,m-1}, \quad m = 1, 2, 3, 4. \tag{C.15}$$

We convert these special payment amounts to rates of pay:

$$spr_h^* = -\frac{Surp_h^*}{\ddot{a}_{\overline{15}|y_h^V} \times TSal_h}, \tag{C.16}$$

$$spr_h = spr_{h-1} + spr_h^*. \tag{C.17}$$

The new contribution rate is then the normal cost rate plus the total special payment rate including the newly established portion:

$$c_h = max(c_h^{NC} + spr_h, 0). \tag{C.18}$$

- If $Surp_h^* > 0$, the actuarial value of plan assets plus future special payments are enough to finance future liabilities, thus we reduce the previously established special payments proportionally by solving for the reduction factor α in equation

$$Surp_h + \alpha \cdot \sum_{m=1}^{4} SP_{h,m} \cdot \ddot{a}_{\overline{15-3m}|y_h^V} = 0. \tag{C.19}$$

We have no additional special payments set up in valuation h:

$$SP_{h,0} = 0, \tag{C.20}$$

and all previously established special payments are adjusted by α:

$$SP_{h,m} = \alpha \cdot SP_{h-1,m-1}, m = 1, 2, 3, 4, \tag{C.21}$$

$$spr_h = \alpha \cdot spr_{h-1}. \tag{C.22}$$

Again, the new contribution rate is:

$$c_h = max(c_h^{NC} + spr_h, 0). \tag{C.23}$$

Appendix D

Recursive Bond Price

As in Ang and Piazzesi (2003), to derive the equations in (6.9), we first note that for a one-period bond:

$$
\begin{aligned}
P_t^{(1)} &= E_t(M_{t+1}) \\
&= E_t[exp(-\delta_0 - \delta_1 x_t - \frac{1}{2}\lambda_t^T P P^T \lambda_t - \lambda_t^T P \epsilon_{t+1})] \\
&= exp(-\delta_0 - \delta_1 x_t - \frac{1}{2}\lambda_t^T P P^T \lambda_t) E_t[exp(-\lambda_t^T P \epsilon_{t+1})]
\end{aligned}
\tag{D.1}
$$

If x is normally distributed with mean μ and variance s^2, its moment generating function is

$$
E[e^{\gamma x}] = exp\{\gamma\mu + \frac{1}{2}\gamma^2 s^2\}
\tag{D.2}
$$

Here $\epsilon_{t+1} \overset{i.i.d}{\sim} N(0, I)$ implies that

$$
\begin{aligned}
P_t^{(1)} &= exp(-\delta_0 - \delta_1 x_t - \frac{1}{2}\lambda_t^T P P^T \lambda_t) exp(\frac{1}{2}\lambda_t^T P P^T \lambda_t) \\
&= exp(-\delta_0 - \delta_1 x_t).
\end{aligned}
\tag{D.3}
$$

Suppose that the price of an n-period bond is given by $P_t^{(n)} = exp(A_n + B_n^T x_t)$. Now we show that the exponential form also applies to the price of $(n+1)$-period bond:

$$
\begin{aligned}
P_t^{(n+1)} &= E_t(M_{t+1} P_{t+1}^{(n)}) \\
&= E_t\Big[exp(-\delta_0 - \delta_1 x_t - \frac{1}{2}\lambda_t^T P P^T \lambda_t - \lambda_t^T P \epsilon_{t+1} + A_n + B_n^T x_{t+1})\Big] \\
&= exp(-\delta_0 - \delta_1 x_t - \frac{1}{2}\lambda_t^T P P^T \lambda_t + A_n) E_t\Big[exp(-\lambda_t^T P \epsilon_{t+1} + B_n^T x_{t+1})\Big] \\
&= exp(-\delta_0 - \delta_1 x_t - \frac{1}{2}\lambda_t^T P P^T \lambda_t + A_n) E_t\Big[exp\{-\lambda_t^T P \epsilon_{t+1} + B_n^T(\nu + \Phi x_t + P\epsilon_{t+1})\}\Big] \\
&= exp(-\delta_0 + A_n + B_n^T \nu + (B_n^T \Phi - \delta_1) x_t - \frac{1}{2}\lambda_t^T P P^T \lambda_t) E_t\Big[exp\{(B_n^T - \lambda_t^T) P \epsilon_{t+1}\}\Big]
\end{aligned}
\tag{D.4}
$$

Again, the normally distributed ϵ_{t+1} implies that

$$\begin{aligned} E_t\Big[exp\{(B_n^T - \lambda_t^T)P\epsilon_{t+1}\}\Big] &= exp[\frac{1}{2}(B_n^T - \lambda_t^T)PP^T(B_n^T - \lambda_t^T)^T] \\ &= exp[\frac{1}{2}B_n^T PP^T B_n - B_n^T PP^T \lambda_t + \frac{1}{2}\lambda_t^T PP^T \lambda_t] \end{aligned} \tag{D.5}$$

Taking (D.5) into (D.4), we get

$$\begin{aligned} P_t^{(n+1)} &= exp(-\delta_0 + A_n + B_n^T\nu + (B_n^T\Phi - \delta_1)x_t + \frac{1}{2}B_n^T PP^T B_n - B_n^T PP^T \lambda_t) \\ &= exp(-\delta_0 + A_n + B_n^T\nu + (B_n^T\Phi - \delta_1)x_t + \frac{1}{2}B_n^T PP^T B_n - B_n^T PP^T(\lambda_0 + \Lambda_1 x_t)) \\ &= exp(-\delta_0 + A_n + B_n^T(\nu - PP^T\lambda_0) + \frac{1}{2}B_n^T PP^T B_n + (B_n^T\Phi - \delta_1 - B_n^T PP^T \Lambda_1)x_t) \end{aligned} \tag{D.6}$$

Matching coefficients, we get

$$\begin{aligned} A_{n+1} &= -\delta_0 + A_n + B_n^T(\nu - PP^T\lambda_0) + \frac{1}{2}B_n^T PP^T B_n \\ B_{n+1}^T &= B_n^T(\Phi - PP^T\Lambda_1) - \delta_1 \end{aligned} \tag{D.7}$$

which is equation (6.9).