An Approach to Constructing "Good" Two-level Orthogonal Factorial Designs with Large Run Sizes

by

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Abstract

Due to the increasing demand for two-level fractional factorials in areas of science and technology, it is highly desirable to have a simple and convenient method available for constructing optimal factorials. Minimum G_2-aberration is a popular criterion to use for selecting optimal designs. However, direct application of this criterion is challenging for large designs. In this project, we propose an approach to constructing a "good" factorial with a large run size using two small minimum G_2-aberration designs. Theoretical results are derived that allow the word length pattern of the large design to be obtained from those of the two small designs. Regular 64-run factorials are used to evaluate this approach. The designs from our approach are very close to the corresponding minimum aberration designs, and they are even equivalent to the corresponding minimum aberration designs, when the number of factors is large.

Keywords: Fractional factorial; Minimum aberration; Minimum G_2-aberration; Word length pattern.

Dedication

To my family!

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Chapter 1

Introduction

Fractional factorials are widely employed in areas of science and technology owing to their flexibility and run size economy. The primary problem for us to use these designs is how to choose an optimal design for a given number of factors and run size. Various criteria have been suggested to deal with this problem, but it seems impractical to provide optimal designs for each criterion. The most popular criterion for comparing regular fractional factorials is minimum aberration, introduced by Fries and Hunter (1980). A generalization of this criterion, called minimum G_2-aberration, was proposed by Tang and Deng (1999) for comparing both regular and nonregular fractional factorials. Although minimum aberration designs are completely known for up to 128 runs, results for more than 128 runs are quite limited. Minimum G_2-aberration designs are available for up to 96 runs and in fact only partially available for more than 32 runs.

This project presents a simple and efficient approach to constructing a "good" large design using two small minimum G_2-aberration designs. We focus on orthogonal factorials with two levels, which can be classified into two categories, regular fractional factorials and nonregular fractional factorials. A regular fractional factorial 2^{m-p}, having m factors of two levels, $m-p$ independent factors, and 2^{m-p} runs, is determined by its defining relation which contains p independent defining words. Regular designs have a property that any two effects are either orthogonal or fully aliased. In contrast to a 2^{m-p} design, a nonregular fractional factorial has some complex aliasing structure, meaning that there exist two partially aliased effects. Tang and Deng (1999) provided a formal definition for both regular and nonregular factorials, which is introduced in Section 1.1.

The rest of this chapter reviews J-characteristics, orthogonal factorials, and the criteria of minimum aberration and minimum G_2-aberration. Chapter 2 proposes our approach to constructing a large design using two smaller designs, and then studies some relationships between the large design and the two small designs. The results are presented in Sections 2.2-2.4. For Chapter 3, the general applications of the results derived in Chapter 2 are discussed in Section 3.1. We then use a specific case to assess the goodness of designs constructed by our approach. We construct regular 64-run factorials with $7 \le m \le 63$ factors using the approach suggested in Section 2.1. For each number m of factors, we choose a design with the least aberration from those constructed designs as the best one, and compare it with the minimum aberration design. Chapter 4 summarizes this project and discusses some possible future work.

1.1 Notation and Background

Suppose design D is an orthogonal fractional factorial (regular or nonregular) with m factors and n runs. For convenience, we write D as a set of m columns, $D = \{d_1, \dots, d_m\}$, or an $n \times m$ matrix, $D = (d_{ij})$, where $d_{ij} \in \{-1, 1\}$. For $1 \le k \le m$ and any k-subset $u = \{d_{j_1}, \dots, d_{j_k}\}$ of D, Deng and Tang (1999) defined

$$J_k(u) = J_k(d_{j_1}, \dots, d_{j_k}) = \left| \sum_{i=1}^{n} d_{ij_1} \cdots d_{ij_k} \right|, \tag{1.1}$$

where d_{ij_1} is the ith component of column d_{j_1}. $J_0(\phi) = n$ was also defined. Here, $J_1(u) = J_2(u) = 0$, as the numbers of the two levels in any column of D are identical, and any two columns are orthogonal in D. Tang and Deng (1999) summarized the $J_k(u)$ values in the following definition.

Definition 1.1.1. The $J_k(u)$ values in (1.1) are called the J-characteristics of design D.

According to (1.1), we have the following fact, which will be used in the proofs of the theorems in Sections 2.2-2.4.

Fact 1.1.1. *For any k-subset $u = \{d_{j_1}, \dots, d_{j_k}\}$ of D and a column $d \in \{d_1, \dots, d_m\}$ in D, we have*

(a)

$$J_{k+1}(d_{j_1}, \dots, d_{j_k}, I_n) = J_k(d_{j_1}, \dots, d_{j_k}),$$

(b)

$$d^h = \underbrace{d \cdots d}_{h} = \begin{cases} I_n & \text{if } h \text{ is even,} \\ d & \text{if } h \text{ is odd,} \end{cases}$$

(c)

$$J_{h+1}(\underbrace{d, \dots, d}_{h}, I_n) = J_h(\underbrace{d, \dots, d}_{h}) = \begin{cases} n & \text{if } h \text{ is even,} \\ 0 & \text{if } h \text{ is odd,} \end{cases}$$

(d)

$$J_{k+h}(d_{j_1}, \dots, d_{j_k}, \underbrace{d, \dots, d}_{h}) = \begin{cases} J_k(d_{j_1}, \dots, d_{j_k}) & \text{if } h \text{ is even,} \\ J_{k+1}(d_{j_1}, \dots, d_{j_k}, d) & \text{if } h \text{ is odd,} \end{cases}$$

where I_n is the identity column of length n with all 1's.

Based on the J-characteristics of D, Deng and Tang (1999) introduced the notion of generalized resolution. Let r be the smallest integer such that $\max_{|u|=r} J_r(u) > 0$, where the maximization is over all the subsets of r distinct columns of D. Then the generalized resolution of design D is defined to be

$$R(D) = r + \left[1 - \max_{|u|=r} J_r(u)/n\right]. \tag{1.2}$$

Clearly, $r \leq R(D) < r+1$. Note that $R(D) \geq 3$ for orthogonal designs, as $J_1(u) = J_2(u) = 0$.

For a regular design D, $J_k(u) = n$ or 0, as effects in u are either fully aliased or orthogonal. But, for a nonregular factorial, there exists a u such that $0 < J_k(u) < n$. According to the values of J-characteristics, Tang and Deng (1999) gave a formal definition for regular and nonregular designs.

Definition 1.1.2. A fractional factorial D is said to be regular if $J_k(u) = n$ or 0 for all $u \subseteq D$. It is said to be nonregular if there exists a $u \subseteq D$ such that $0 < J_k(u) < n$.

It is clear that the defining relation of a regular design D is the collection of all subsets u's such that $J_k(u) = n$ for $k = 1, \ldots, m$. This means that if there exists a k such that $J_k(u) = n$, then a word in the defining relation is formed by those k columns in u. Let $A_k(D)$ be the number of words of length k in the defining relation, and then the word length pattern of design D is defined as the vector, $W(D) = (A_1(D), A_2(D), A_3(D), \ldots, A_m(D))$. Obviously, $A_1(D) = A_2(D) = 0$.

For two regular factorials D_1 and D_2 with the same number of factors and run size, the minimum aberration is utilized to compare them. Let r be the smallest integer such that $A_r(D_1) \neq A_r(D_2)$. If $A_r(D_1) < A_r(D_2)$, then D_1 is said to have less aberration than D_2. If no design has less aberration than D_1, then we say that D_1 has *minimum aberration.*

The minimum G_2-aberration, a generalization of the minimum aberration, is used to assess the goodness of general fractional factorials. For a design D, let

$$B_k(D) = \frac{1}{n^2} \sum_{|u|=k} J_k^2(u) = \frac{1}{n^2} \sum_{1 \leq j_1 < \cdots < j_k \leq m} J_k^2(d_{j_1}, \ldots, d_{j_k}), \text{ for } 1 \leq k \leq m, \tag{1.3}$$

where $B_1(D) = B_2(D) = 0$. For two factorials D_1 and D_2, let r be the smallest integer such that $B_r(D_1) \neq B_r(D_2)$. If $B_r(D_1) < B_r(D_2)$, then D_1 is said to have less G_2-aberration than D_2. If no design has less G_2-aberration than D_1, then we say that D_1 has *minimum G_2-aberration.* If D is a regular factorial, then $B_k(D) = A_k(D)$, which implies that minimum G_2-aberration is equivalent to minimum aberration for regular designs.

For $1 \le k \le m$, Butler (2003) defined

$$M_k(D) = \frac{1}{n^2} \sum_{j_1=1}^{m} \cdots \sum_{j_k=1}^{m} J_k^2(d_{j_1}, \ldots, d_{j_k}), \text{ for } 1 \le k \le m. \tag{1.4}$$

Clearly, $M_k(D)$ is greater than $B_k(D)$, as it considers all permutations for each collection of k columns and also allows columns to occur in $\{d_{j_1}, \ldots, d_{j_k}\}$ more than once. The quantity $M_k(D)$ is instrumental in determining the constants in the theorems of Sections 2.3 and 2.4.

Chapter 2

General Theoretical Results

2.1 Constructing A Large Design Using Two Small Designs

We firstly review some basic definitions. Let $x = (x_1, \ldots, x_{n_1})^T$ and $y = (y_1, \ldots, y_{n_2})^T$. The Kronecker product of two vectors x and y is defined as

$$x \otimes y = (x_1 y_1, \ldots, x_1 y_{n_2}, \ldots, x_{n_1} y_1, \ldots, x_{n_1} y_{n_2})^T.$$

Tang (2006) provided a simple way to calculate the J-characteristic of Kronecker products.

Lemma 2.1.1. *We have that*

$$J(a_1 \otimes b_1, \ldots, a_k \otimes b_k) = J(a_1, \ldots, a_k) J(b_1, \ldots, b_k),$$

where $a_j = (a_{1j}, \ldots, a_{n_1 j})^T$ *and* $b_j = (b_{1j}, \ldots, b_{n_2 j})^T$ *for* $j = 1, \ldots, k$.

Let D_1 and D_2 be two factorials, either regular or nonregular. Design D_1 is an n_1-run design with m_1 factors and can be expressed by a set of m_1 columns, $D_1 = \{a_1, \ldots, a_{m_1}\}$. Similarly, D_2 has m_2 factors and n_2 runs, and can be written as a set of m_2 columns, $D_2 = \{b_1, \ldots, b_{m_2}\}$. A large design D can then be obtained by taking the Kronecker product of two designs D_1 and D_2,

$$\begin{aligned} D &= D_1 \otimes D_2 \\ &= \{a_1, \ldots, a_{m_1}\} \otimes \{b_1, \ldots, b_{m_2}\} \\ &= \{a_1 \otimes b_1, \ldots, a_1 \otimes b_{m_2}, \ldots, a_{m_1} \otimes b_1, \ldots, a_{m_1} \otimes b_{m_2}\}. \end{aligned} \tag{2.1}$$

Clearly, design D is a factorial with m factors and n runs, where $m = m_1 m_2$ and $n = n_1 n_2$. Moreover, D is a regular factorial if both D_1 and D_2 are regular. Otherwise, D is a nonregular factorial.

For any k-subset $u = \{a_{i_1} \otimes b_{j_1}, \ldots, a_{i_k} \otimes b_{j_k}\}$, where $1 \le k \le m$, by Lemma 2.1.1, we have

$$\begin{aligned} J_k(u) &= J_k\left(a_{i_1} \otimes b_{j_1}, \ldots, a_{i_k} \otimes b_{j_k}\right) \\ &= J_k\left(a_{i_1}, \ldots, a_{i_k}\right) J_k\left(b_{j_1}, \ldots, b_{j_k}\right). \end{aligned} \tag{2.2}$$

Here, since each column in D is the Kronecker product of a column from D_1 and one from D_2 and the total number of columns in D is $m_1 m_2$, for any $a_i \in \{a_1, \ldots, a_{m_1}\}$ and $b_j \in \{b_1, \ldots, b_{m_2}\}$, the numbers of a_i and b_j contributing to D are m_1 and m_2, respectively. Hence, for an arbitrary k-subset $u = \{a_{i_1} \otimes b_{j_1}, \ldots, a_{i_k} \otimes b_{j_k}\}$ with $1 \le k \le m$, each of the collections $\{a_{i_1}, \ldots, a_{i_k}\}$ and $\{b_{j_1}, \ldots, b_{j_k}\}$ in (2.2) may contain some repeated columns. This observation is important for the proofs in the following sections of this chapter.

2.2 Doubling

Doubling is a special case of the construction in (2.1), where

$$D_1 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{2.3}$$

Here, design D_1 only has two columns $a_1 = (1,1)^T$ and $a_2 = (-1,1)^T$. Design D_2 is an ordinary factorial, either regular or nonregular, with m_2 factors and n_2 runs. The double of D_2 is then defined as

$$\begin{aligned} D &= D_1 \otimes D_2 \\ &= \begin{bmatrix} D_2 & -D_2 \\ D_2 & D_2 \end{bmatrix}. \end{aligned} \tag{2.4}$$

Clearly, design D has $m = 2m_2$ factors and $n = 2n_2$ runs. If D_2 is regular, Chen and Cheng (2006) derived a relationship between the word-length pattern of D_2 and that of its corresponding D, which is given in Theorem 2.2.1. The same relationship also holds for nonregular designs but it does require a new proof.

Theorem 2.2.1. *Suppose k is a positive integer with $1 \le k \le 2m_2$. Then*

$$B_k(D) = \begin{cases} 0 & \text{if } k \in \{1, 2\}, \\ \displaystyle\sum_{t=0}^{\lfloor (k-3)/2 \rfloor} 2^{k-2t-1} \binom{m_2 - (k-2t)}{t} B_{k-2t}(D_2) + \binom{m_2}{k/2} & \text{if } k \text{ is a multiple of } 4, \\ \displaystyle\sum_{t=0}^{\lfloor (k-3)/2 \rfloor} 2^{k-2t-1} \binom{m_2 - (k-2t)}{t} B_{k-2t}(D_2) & \text{otherwise,} \end{cases} \tag{2.5}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Proof. It is obvious that $B_1(D) = B_2(D) = 0$, as $J_1 = J_2 = 0$ for any orthogonal design D. For $3 \le k \le 2m_2$, since there are only 2 columns in D_1, each column in $\{b_{j_1}, \ldots, b_{j_k}\}$ occurs at most twice when we calculate $J_k(u)$ in (2.2). Let t be the number of columns occurring exactly twice in $\{b_{j_1}, \ldots, b_{j_k}\}$. We have that the remaining $(k-2t)$ columns in $\{b_{j_1}, \ldots, b_{j_k}\}$, different from the above t columns, are all distinct. This implies that there are $\binom{m_2-(k-2t)}{t}$ ways to choose the repeating columns for each $\{b_{j_1}, \ldots, b_{j_k}\}$. By part (d) of Fact 1.1.1, we then have that $J_k(b_{j_1}, \ldots, b_{j_k}) = \binom{m_2-(k-2t)}{t} J_{k-2t}(b_{j_1}, \ldots, b_{j_{k-2t}})$. On the other hand, design D_1 in (2.3) gives $J_k(a_{i_1}, \ldots, a_{i_k}) = 0$ or 2, and $J_k(a_{i_1}, \ldots, a_{i_k}) = 2$, only if the number of $a_2 = (-1,1)^T$ in $\{a_{i_1}, \ldots, a_{i_k}\}$ is even. As there are t columns occurring twice in $\{b_{j_1}, \ldots, b_{j_k}\}$, column a_2 is included in $\{a_{i_1}, \ldots, a_{i_k}\}$ at least t times. We then study two situations, (a) t is even and (b) t is odd, although the final results are the same for both situations. For situation (a), since t is even, the number of a_2's which correspond to $k-2t$ distinct columns in $\{b_{j_1}, \ldots, b_{j_k}\}$ is even for otherwise $J_k(u) = 0$. Then the total number of ways to choose the number of a_2's is $\binom{k-2t}{0} + \binom{k-2t}{2} + \binom{k-2t}{4} + \cdots = 2^{k-2t}/2$. Resembling situation (a), the number of a_2's which correspond to $k-2t$ distinct columns is odd, as t is odd in situation (b). Then there are $\binom{k-2t}{1} + \binom{k-2t}{3} + \binom{k-2t}{5} + \cdots = 2^{k-2t}/2$ ways to choose the number of a_2's. As $J_1 = J_2 = 0$ for any orthogonal design D, t must satisfy $k - 2t \ge 3$ in order to have $J_{k-2t}(b_{j_1}, \ldots, b_{j_{k-2t}}) > 0$. Since t is a nonnegative integer, we have $t \in [0, (k-3)/2]$, if k is odd, and $t \in [0, (k-4)/2]$, if k is even. We have two cases: (i) k is odd, and (ii) k is even. For any k-subset $u = \{a_{i_1} \otimes b_{j_1}, \ldots, a_{i_k} \otimes b_{j_k}\}$, where $3 \le k \le 2m_2$, case (i) gives

$$\begin{aligned}
B_k(D) =& \frac{1}{(2n_2)^2} \sum_{|u|=k} J_k^2(u) \\
=& \frac{1}{(2n_2)^2} \sum_{|u|=k} J_k^2(a_{i_1}, \ldots, a_{i_k}) J_k^2(b_{j_1}, \ldots, b_{j_k}) \\
=& \sum_{t=0}^{(k-3)/2} 2^{k-2t}/2 \binom{m_2-(k-2t)}{t} \frac{1}{n_2^2} \sum_{1 \le j_1 < \cdots < j_{k-2t} \le m_2} J_{k-2t}^2(b_{j_1}, \ldots, b_{j_{k-2t}}) \\
=& \sum_{t=0}^{(k-3)/2} 2^{k-2t}/2 \binom{m_2-(k-2t)}{t} B_{k-2t}(D_2),
\end{aligned} \tag{2.6}$$

For case (ii), except for a similar term like (2.6), $B_k(D)$ contains an extra constant term for an even $k/2$, as there exists $t = k/2$ under (ii) such that $k - 2t = 0$. This implies that there are $k/2$ distinct columns in $\{b_{j_1}, \ldots, b_{j_k}\}$, each column occurring twice. We then have that the number of a_2's in $\{a_{i_1}, \ldots, a_{i_k}\}$ is exact $k/2$. Since $J_k(b_{j_1}, \ldots, b_{j_k}) = n_2$ by part (c) of Fact 1.1.1, and $J_k(a_{i_1}, \ldots, a_{i_k}) = n_1 = 2 > 0$ only when $k/2$ is even, a non-zero constant term exists only when $k/2$ is even. The number of ways to choose those $k/2$ columns is $\binom{m_2}{k/2}$. The final result for an even $k/2$ is

$$B_k(D) = \sum_{t=0}^{(k-4)/2} 2^{k-2t}/2 \binom{m_2-(k-2t)}{t} B_{k-2t}(D_2) + \binom{m_2}{k/2}. \tag{2.7}$$

Combining (2.6) and (2.7), Theorem 2.2.1 is obtained. □

Using Theorem 2.2.1, detailed expressions for $B_3(D)$, $B_4(D)$ and $B_5(D)$ are easily obtained, which are shown below

$$B_3(D) = 4B_3(D_2), \tag{2.8}$$

$$B_4(D) = 8B_4(D_2) + \binom{m_2}{2}, \tag{2.9}$$

and

$$B_5(D) = 16B_5(D_2) + 4(m_2 - 3)B_3(D_2). \tag{2.10}$$

These equations will be used in the next chapter.

2.3 Small Design Without A Column of 1's

Design D in Section 2.2 has a restriction in that it can only double the number m_2 of factors and the run size n_2 of the original design D_2. This implies that the number m of factors and the run size n of D are fixed for given m_2 and n_2 of D_2. To make m and n more flexible, we construct D by using a general design D_1. Design D_1 discussed in this section is any factorial, with the only requirement that it does not contain a column of all 1's. Note that we always assume D_2 does not have a column of all 1's.

As D_1 is relaxed, $B_k(D)$ is also related to $B_s(D_1)$ for $s \leq k$. A relationship can be established among $B_s(D_1)$, $B_t(D_2)$ and $B_k(D)$ for $s, t \leq k$, which is shown in the following theorem.

Theorem 2.3.1. *Suppose k is a positive integer with $1 \leq k \leq m_1 m_2$. Then*

$$B_k(D) = \begin{cases} 0 & \text{if } k \in \{1,2\}, \\ \sum_{s=0}^{l}\sum_{t=0}^{l} C_{st} B_{k-2s}(D_1) B_{k-2t}(D_2) & \text{if } k \geq 3 \text{ is odd}, \\ \sum_{s=0}^{l}\sum_{t=0}^{l} C_{st} B_{k-2s}(D_1) B_{k-2t}(D_2) + \sum_{s=0}^{l} C_s^{(1)} B_{k-2s}(D_1) + \sum_{t=0}^{l} C_t^{(2)} B_{k-2t}(D_2) + C & \text{if } k \geq 3 \text{ is even}, \end{cases} \tag{2.11}$$

where $l = \lfloor (k-3)/2 \rfloor$, and $C_{00} > 0$. All constants in (2.11) depend on m_1, m_2, n_1, n_2, not on choices of D_1 and D_2 for given m_1, m_2, n_1, n_2.

$$\begin{array}{ccc} B_k(D_1) \text{ and } B_k(D_2) & \longleftrightarrow & B_k(D) \\ \updownarrow \text{I} & & \updownarrow \text{III} \\ M_k(D_1) \text{ and } M_k(D_2) & \overset{\text{II}}{\longleftrightarrow} & M_k(D) \end{array}$$

Figure 2.1: Road Map for the Proof

Proof. Obviously, $B_1(D) = B_2(D) = 0$. For $3 \le k \le m_1 m_2$, instead of directly studying a relationship of B_k's among designs D_1, D_2 and D, we separate it into three steps I, II, and III, which is shown in Figure 2.1. Step I connects $B_k(D_1)$ and $B_k(D_2)$ with $M_k(D_1)$ and $M_k(D_2)$, respectively. Step II links $M_k(D_1)$ and $M_k(D_2)$ with $M_k(D)$. Since M_k considers all permutations for each collection of k columns and allows the columns occurring more than once in each collection, the relationship in Step II is straightforward and given by

$$M_k(D) = M_k(D_1)M_k(D_2), \text{ for } 1 \le k \le m_1 m_2. \tag{2.12}$$

The last step gives a connection between $M_k(D)$ and $B_k(D)$. In practice, two relationships in Step I are identical and the inverse of this relationship is what Step III needs, as D_1, D_2 and D are general factorials without a column of 1's. Hence, we choose design D_2 to study this relationship. Let t^* be the number of columns occurring more than once in $\{b_{j_1}, \ldots, b_{j_k}\}$, say columns $b_{j_{k-t^*+1}}, \ldots, b_{j_k}$, and h_g be the number of b_{j_g}'s in $\{b_{j_1}, \ldots, b_{j_k}\}$ for $k - t^* + 1 \le g \le k$. We then consider two situations, (i) all h_g's are even, and (ii) at least one h_g is odd. Situation (i) gives that there exists a t such that $\sum_{g=k-t^*+1}^{k} h_g = 2t$. We then have $b_{j_{k-t^*+1}}^{h_{k-t^*+1}} \cdots b_{j_k}^{h_k} = I_{n_2}$ by part (d) of Fact 1.1.1, which implies that $J_k\left(b_{j_1}, \ldots, b_{j_k}\right) = J_{k-2t}\left(b_{j_1}, \ldots, b_{j_{k-2t}}\right)$. For situation (ii), suppose $h_{k-t^*+1}, \ldots, h_{g'}$ are odd, where g' can be any number in $\{k - t^* + 1, \ldots, k\}$. Then there exists a t such that $\sum_{g=k-t^*+1}^{g'} (h_g - 1) + \sum_{g'}^{k} h_g = \sum_{g=k-t^*+1}^{k} h_g - (g' - k + t^*) = 2t$. By part (b) of Fact 1.1.1, we then obtain $b_{j_{k-t^*+1}}^{h_{k-t^*+1}} \cdots b_{j_k}^{h_k} = b_{j_{k-t^*+1}} \cdots b_{j_{g'}}$, and $J_k\left(b_{j_1}, \ldots, b_{j_k}\right) = J_{k-2t}\left(b_{j_1}, \ldots, b_{j_{k-\sum h_g}}, b_{j_{k-t^*+1}}, \ldots, b_{j_{g'}}\right) = J_{k-2t}\left(b_{j_1}, \ldots, b_{j_{k-2t}}\right)$. Both situations (i) and (ii) give that J_k are related to J_{k-2t}'s with $3 \le k - 2t \le k$. We consider two cases. Case (1): k is odd. We then have

$$\begin{aligned}
M_k(D_2) &= \frac{1}{n_2^2} \sum_{j_1=1}^{m} \cdots \sum_{j_k=1}^{m} J_k^2(b_{j_1}, \ldots, b_{j_k}) \\
&= \sum_{t=0}^{(k-3)/2} C_t^{*(2)} \sum_{1 \le j_1 < \cdots < j_{k-2t} \le m_2} \frac{1}{n_2^2} J_{k-2t}^2\left(b_{j_1}, \ldots, b_{j_{k-2t}}\right) \\
&= \sum_{t=0}^{(k-3)/2} C_t^{*(2)} B_{k-2t}(D_2),
\end{aligned} \tag{2.13}$$

where $C_t^{*(2)}$ is a positive constant, as $C_t^{*(2)}$ is the product of two numbers, the number of ways to choose the repeating columns and that of ways to permute k columns in each collection. Similarly, for D_1 and D, we have

$$M_k(D_1) = \sum_{s=0}^{(k-3)/2} C_s^{*(1)} B_{k-2s}(D_1), \tag{2.14}$$

$$M_k(D) = \sum_{v=0}^{(k-3)/2} C_v^{*(3)} B_{k-2v}(D), \tag{2.15}$$

where $C_s^{*(1)}$ and $C_v^{*(3)}$ are positive constants for $0 \le s, v \le (k-3)/2$. Using (2.12), a recursion formula for an odd k is obtained

$$\begin{aligned}\sum_{v=0}^{(k-3)/2} C_v^{*(3)} B_{k-2v}(D) &= \sum_{s=0}^{(k-3)/2} C_s^{*(1)} B_{k-2s}(D_1) \sum_{t=0}^{(k-3)/2} C_t^{*(2)} B_{k-2t}(D_2) \\ &= \sum_{s=0}^{(k-3)/2} \sum_{t=0}^{(k-3)/2} C_{st}^{*} B_{k-2s}(D_1) B_{k-2t}(D_2). \end{aligned} \tag{2.16}$$

A formula for $B_k(D)$ can then be obtained from the above recursion formula,

$$B_k(D) = \sum_{s=0}^{(k-3)/2} \sum_{t=0}^{(k-3)/2} C_{st} B_{k-2s}(D_1) B_{k-2t}(D_2), \tag{2.17}$$

where C_{00} is a positive coefficient for the leading term $B_k(D_1)B_k(D_2)$ for $3 \le k \le m_1 m_2$, as can be verified by induction. For $k = 3$, using (2.16), it is obvious that $B_3(D) = \frac{C_{00}^*}{C_0^{*(3)}} B_3(D_1) B_3(D_2)$ with $\frac{C_{00}^*}{C_0^{*(3)}} > 0$. For $k > 3$, let k^* be the maximum odd number in the range $[3, m_1 m_2]$. Suppose that (2.17) with a positive C_{00} is true for $k \le (k^* - 2)$. For $k = k^*$, using (2.16), we have

$$\begin{aligned}
C_0^{*(3)} B_{k^*}(D) &= \sum_{s=0}^{(k^*-3)/2} \sum_{t=0}^{(k^*-3)/2} C_{st}^{*} B_{k^*-2s}(D_1) B_{k^*-2t}(D_2) - \sum_{v=1}^{(k^*-3)/2} C_v^{*(3)} B_{k^*-2v}(D) \\
&= C_{00}^{*} B_{k^*}(D_1) B_{k^*}(D_2) \\
&\quad + \sum_{t=0}^{(k^*-3)/2} C_{0t}^{*} B_{k^*}(D_1) B_{k^*-2t}(D_2) + \sum_{s=0}^{(k^*-3)/2} C_{s0}^{*} B_{k^*-2s}(D_1) B_{k^*}(D_2) \\
&\quad + \sum_{s=1}^{(k^*-3)/2} \sum_{t=1}^{(k^*-3)/2} C_{st}^{*} B_{k^*-2s}(D_1) B_{k^*-2t}(D_2) - \sum_{v=1}^{(k^*-3)/2} C_v^{*(3)} B_{k^*-2v}(D) \\
&= C_{00}^{*} B_{k^*}(D_1) B_{k^*}(D_2) + \sum_{t=0}^{(k^*-3)/2} C_{0t}^{*} B_{k^*}(D_1) B_{k^*-2t}(D_2) \\
&\quad + \sum_{s=0}^{(k^*-3)/2} C_{s0}^{*} B_{k^*-2s}(D_1) B_{k^*}(D_2) + \sum_{s=1}^{(k^*-3)/2} \sum_{t=1}^{(k^*-3)/2} C_{st}^{*} B_{k^*-2s}(D_1) B_{k^*-2t}(D_2) \\
&\quad - \sum_{v=1}^{(k^*-3)/2} C_v^{*(3)} \sum_{s=0}^{(k^*-2v-3)/2} \sum_{t=0}^{(k^*-2v-3)/2} C_{st}^{*} B_{k^*-2v-2s}(D_1) B_{k^*-2v-2t}(D_2) \\
&= C_{00}^{*} B_{k^*}(D_1) B_{k^*}(D_2) + \sum_{t=0}^{(k^*-3)/2} C_{0t}^{*} B_{k^*}(D_1) B_{k^*-2t}(D_2) \\
&\quad + \sum_{s=0}^{(k^*-3)/2} C_{s0}^{*} B_{k^*-2s}(D_1) B_{k^*}(D_2) + \sum_{s=1}^{(k^*-3)/2} \sum_{t=1}^{(k^*-3)/2} C_{st}^{**} B_{k^*-2s}(D_1) B_{k^*-2t}(D_2) \\
&= \sum_{s=0}^{(k^*-3)/2} \sum_{t=0}^{(k^*-3)/2} C_{st}^{**} B_{k^*-2s}(D_1) B_{k^*-2t}(D_2),
\end{aligned} \tag{2.18}$$

where $C_{00}^{**} = C_{00}^{*} > 0$. As the coefficient $C_0^{*(3)}$ of $B_k(D)$ is positive, equation (2.17) is true for $3 \le k \le k^*$. Case (2): k is even. There exist some t^*'s such that $\sum_{g=k-t^*+1}^{k} h_g = 2t = k$, which implies that $J_k(b_{j_1}, \ldots, b_{j_k}) = n_2$, only if all h_g's are even. Let m^* be the number of values of t^*, we then obtain

$$\begin{aligned} M_k(D_2) &= \sum_{t=0}^{(k-4)/2} C_t^{*(2)} B_{k-2t}(D_2) + \sum_{t^*=1}^{m^*} C'_{2_{t^*}}, \\ &= \sum_{t=0}^{(k-4)/2} C_t^{*(2)} B_{k-2t}(D_2) + C'_2, \end{aligned} \tag{2.19}$$

where $C'_{2_{t^*}}$ is the number of ways to choose the t^* columns. Similar to the case of odd k, $M_k(D_1)$ and $M_k(D)$ can be easily written as

$$M_k(D_1) = \sum_{s=0}^{(k-4)/2} C_s^{*(1)} B_{k-2s}(D_1) + C'_1, \tag{2.20}$$

$$M_k(D) = \sum_{v=0}^{(k-4)/2} C_v^{*(3)} B_{k-2v}(D) + C', \tag{2.21}$$

where $C_s^{*(1)}$ and $C_v^{*(3)}$ are positive constants. A recursive formula is derived using (2.12),

$$\sum_{v=0}^{(k-4)/2} C_v^{*(3)} B_{k-2v}(D) + C' = \left(\sum_{s=0}^{(k-4)/2} C_s^{*(1)} B_{k-2s}(D_1) + C'_1 \right) \left(\sum_{t=0}^{(k-4)/2} C_t^{*(2)} B_{k-2t}(D_2) + C'_2 \right).$$

By induction, the formula for each $B_k(D)$ with an even k is

$$B_k(D) = \sum_{s=0}^{\frac{k-4}{2}} \sum_{t=0}^{\frac{k-4}{2}} C_{st} B_{k-2s}(D_1) B_{k-2t}(D_2) + \sum_{s=0}^{\frac{k-4}{2}} C_s^{(1)} B_{k-2s}(D_1) + \sum_{t=0}^{\frac{k-4}{2}} C_t^{(2)} B_{k-2t}(D_2) + C, \tag{2.22}$$

where $C_{00} > 0$. We then arrive at (2.11) with a positive coefficient of the leading term $B_k(D_1)B_k(D_2)$ for $3 \le k \le m_1 m_2$.

□

Obviously, it is hard to determine the constants in (2.11) explicitly for all s and t. Here, we only focus on the cases of $k = 3, 4, 5$, as they are useful in Chapter 3. The simplest case is $k = 3$. It is evident that

$$M_3(D_1) = 3! B_3(D_1),$$

as M_3 considers the permutations of columns in $\{a_{i_1}, a_{i_2}, a_{i_3}\}$. Similarly, we have

$$M_3(D_2) = 3! B_3(D_2), \tag{2.23}$$

and

$$M_3(D) = 3! B_3(D). \tag{2.24}$$

Using (2.12) we obtain

$$B_3(D) = 3!B_3(D_1)B_3(D_2). \tag{2.25}$$

For $k = 4$, by equation (2.20), we have

$$\begin{aligned} M_4(D_1) &= C_0^{*(1)} B_4(D_1) + C_1' \\ &= 4!B_4(D_1) + \left(\binom{4}{2}\binom{m_1}{2} + \binom{m_1}{1} \right) \\ &= 4!B_4(D_1) + 6\binom{m_1}{2} + m_1, \end{aligned} \tag{2.26}$$

where 4! is the number of ways to permute four distinct columns in $\{a_{i_1}, a_{i_2}, a_{i_3}, a_{i_4}\}$, $\binom{4}{2}\binom{m_1}{2}$ is the product of two numbers, the number of ways to choose two distinct columns and that to permute four columns, and $\binom{m_1}{1}$ is the number of ways to choose one column. Similarly for D_2 and D, we obtain

$$M_4(D_2) = 4!B_4(D_2) + 6\binom{m_2}{2} + m_2, \tag{2.27}$$

and

$$M_4(D) = 4!B_4(D) + 6\binom{m}{2} + m. \tag{2.28}$$

Again using (2.12) and after some algebraic calculation, we derive that

$$\begin{aligned} B_4(D) =& 4!B_4(D_1)B_4(D_2) + \left(m_1 + 6\binom{m_1}{2}\right) B_4(D_2) \\ &+ \left(m_2 + 6\binom{m_2}{2}\right) B_4(D_1) + \binom{m_1}{2}\binom{m_2}{2}. \end{aligned} \tag{2.29}$$

Clearly, using equation (2.14) with $k = 5$, we get

$$M_5(D_1) = C_0^{*(1)} B_5(D_1) + C_1^{*(1)} B_3(D_1) = 5!B_5(D_1) + C_1^{*(1)} B_3(D_1),$$

where 5! is the number of the ways to permute five distinct columns in each subset. For the second term in the above equation, it is obvious that there must be only one column occurring at least twice, say columns $a_{i_4} = a_{i_5}$. Two situations are possible: (i) there exists a h such that $a_{i_4} = a_{i_h}$, where $h = 1, 2, 3$, and (ii) $a_{i_4} \neq a_{i_h}$ for $h = 1, 2, 3$. For situation (i) , there are $\binom{3}{1}$ ways to choose a column from $\{a_{i_1}, a_{i_2}, a_{i_3}\}$, and $\binom{5}{3}\binom{2}{1}$ ways to permute those five columns. Situation (ii) has $\binom{m_1-3}{1}$ ways to choose a column from the set without columns a_{i_1}, a_{i_2}, and a_{i_3}, and has $\binom{5}{2}3!$ ways to permute those five columns. We thus have

$$M_5(D_1) = 5!B_5(D_1) + \left(\binom{3}{1}\binom{5}{3}\binom{2}{1} + \binom{m_1-3}{1}\binom{5}{2}3!\right)B_3(D_1)$$
$$= 5!B_5(D_1) + 60(m_1-2)B_3(D_1). \tag{2.30}$$

Obviously, $M_5(D_2)$ and $M_5(D)$ are

$$M_5(D_2) = 5!B_5(D_2) + 60(m_2-2)B_3(D_2), \tag{2.31}$$

and

$$M_5(D) = 5!B_5(D) + 60(m-2)B_3(D). \tag{2.32}$$

Finally, some simple algebra leads to

$$B_5(D) = 5!B_5(D_1)B_5(D_2) + 60(m_2-2)B_5(D_1)B_3(D_2) + 60(m_1-2)B_3(D_1)B_5(D_2)$$
$$+ (27m_1m_2 - 60m_1 - 60m_2 + 126)B_3(D_1)B_3(D_2). \tag{2.33}$$

2.4 Small Design with A Column of 1's

Design D_1 discussed in this section contains one column of 1's which is the identity column I_{n_1}. This implies that design D constructed by such a D_1 has extra m_2 columns. We also derive a similar relationship among $B_s(D_1)$, $B_t(D_2)$, and $B_k(D)$ for $s, t \leq k$, which is shown in the following theorem.

Theorem 2.4.1. *Suppose k is a positive integer with $1 \leq k \leq m_1m_2$. Then*

$$B_k(D) = \begin{cases} 0 & \text{if } k \in \{1,2\}, \\ \sum_{s=0}^{k-3}\sum_{t=0}^{l} C_{st}B_{k-s}(D_1)B_{k-2t}(D_2) + \sum_{t=0}^{l} C_t^{(2)}B_{k-2t}(D_2) & \text{if } k \geq 3 \text{ is odd,} \\ \sum_{s=0}^{k-3}\sum_{t=0}^{l} C_{st}B_{k-s}(D_1)B_{k-2t}(D_2) + \sum_{t=0}^{l} C_t^{(2)}B_{k-2t}(D_2) + \sum_{s=0}^{k-3} C_s^{(1)}B_{k-s}(D_1) + C & \text{if } k \geq 3 \text{ is even,} \end{cases} \tag{2.34}$$

where $l = \lfloor (k-3)/2 \rfloor$, and $C_{00} > 0$. All constants in (2.34) depend on m_1, m_2, n_1, n_2, not on choices of D_1 and D_2 for given m_1, m_2, n_1, n_2.

Proof. This proof is similar to that of Theorem 2.3.1. As D_1 contains column I_{n_1}, $M_k(D_1)$ depends on all $B_{k-s}(D_1)$ for $0 \leq s \leq k-3$. A constant term always exists, regardless of the parity of k, because of a special collection with length k, $\{I_{n_1}, \ldots, I_{n_1}\}$. We then have

$$M_k(D_1) = \sum_{s=0}^{(k-3)} C_s^{*(1)}B_{k-s}(D_1) + C^*, \tag{2.35}$$

where $C_s^{*(1)}$ is a positive constant, as it is the product of two numbers, the number of ways to choosing the repeating columns and that of ways to permute columns in the corresponding collection. Since D_2 and D are two ordinary orthogonal designs, the expressions for M_k derived in the proof of Theorem 2.3.1 are still valid. $M_k(D_2)$ and $M_k(D)$ with an odd k are shown in (2.13) and (2.15) with positive coefficients, respectively. For an even k, equations (2.19) and (2.21) with positive coefficients are also the expressions for $M_k(D_2)$ and $M_k(D)$ in this section, respectively. Using (2.12), recursive formulas are

$$\begin{cases} \sum_{v=0}^{(k-3)/2} C_v^{*(3)} B_{k-2v}(D) = \left(\sum_{s=0}^{k-3} C_s^{*(1)} B_{k-s}(D_1) + C^* \right) \sum_{t=0}^{(k-3)/2} C_t^{*(2)} B_{k-2t}(D_2) & \text{if } k \text{ is odd,} \\ \sum_{v=0}^{(k-4)/2} C_v^{*(3)} B_{k-2v}(D) + C' = \left(\sum_{s=0}^{k-3} C_s^{*(1)} B_{k-s}(D_1) + C^* \right) \left(\sum_{t=0}^{(k-4)/2} C_t^{*(2)} B_{k-2t}(D_2) + C_2' \right) & \text{if } k \text{ is even.} \end{cases}$$

Using induction for odd k's and even k's respectively, we obtain

$$B_k(D) = \begin{cases} \sum_{s=0}^{k-3} \sum_{t=0}^{(k-3)/2} C_{st} B_{k-s}(D_1) B_{k-2t}(D_2) + \sum_{t=0}^{(k-3)/2} C_t^{(2)} B_{k-2t}(D_2) & \text{if } k \text{ is odd,} \\ \sum_{s=0}^{k-3} \sum_{t=0}^{(k-4)/2} C_{st} B_{k-s}(D_1) B_{k-2t}(D_2) + \sum_{t=0}^{(k-4)/2} C_t^{(2)} B_{k-2t}(D_2) + \sum_{s=0}^{k-3} C_s^{(1)} B_{k-s}(D_1) + C & \text{if } k \text{ is even,} \end{cases}$$

where $C_{00} > 0$ for both odd k's and even k's. □

Here, we also determine the constants in $B_k(D)$ using M_k of the corresponding designs D_1, D_2 and D for $k = 3, 4, 5$. Clearly, $M_k(D_2)$ and $M_k(D)$ for $k = 3, 4, 5$ were given in Section 2.3, as D_2 and D are factorials without one column of 1's. For design D_1 with one column of 1's, $M_k(D_1)$ with $k = 3, 4, 5$ are derived as follows.

For $k = 3$, using (2.35), we have

$$M_3(D_1) = C_0^{*(1)} B_3(D_1) + C^*,$$

where $C_0^{*(1)} = 3!$, as the number of ways to permute three distinct columns in each subset is 3!. The term C^* considers the collection $\{a_{i_1}, a_{i_2}, a_{i_3}\}$ with repeating columns. As there are only three columns in each collection, two situations should be studied, (i) a column occurs twice, and (ii) a column occurs 3 times. For situation (i), $J_3^2(a_{i_1}, a_{i_2}, a_{i_3}) > 0$, only when each collection $\{a_{i_1}, a_{i_2}, a_{i_3}\}$ contains one column I_{n_1} apart from two identical columns. There are $(m_1 - 1)$ ways to choose the identical column and $\binom{3}{1}$ ways to permute those three columns. For situation (ii), $J_3^2(a_{i_1}, a_{i_2}, a_{i_3}) > 0$, only when each collection $\{a_{i_1}, a_{i_2}, a_{i_3}\}$ contains three I_{n_1}'s.

We then obtain

$$M_3(D_1) = 3!B_3(D_1) + \left(\binom{3}{1}(m_1 - 1) + 1\right)$$
$$= 3!B_3(D_1) + 3(m_1 - 1) + 1.$$

Combining this equation with (2.23) and (2.24), we have

$$B_3(D) = 3!B_3(D_1)B_3(D_2) + (3m_1 - 2)B_3(D_2). \tag{2.36}$$

For $k = 4$, we obtain

$$M_4(D_1) = C_0^{*(1)}B_4(D_1) + C_1^{*(1)}B_3(D_1) + C^*.$$

Obviously, $C_0^{*(1)} = 4!$. As each collection of $B_3(D_1)$ contains one column I_{n_1} which is different from the other three distinct columns, there are 4! ways to permute those four columns. The last term is from the collections which have the repeating columns. Let h be the number of columns occurring more than once. Clearly, $h = 1, 2$. For $h = 1$, $J_4^2(a_{i_1}, \ldots, a_{i_4}) > 0$, only when $a_{i_1} = a_{i_g}$ for $g = 2, 3, 4$. There are m_1 distinct columns which can be used to form such collections. For $h = 2$, since there are four columns in each collection, each column occurs exactly twice. the number of ways to choose two distinct columns is $\binom{m_1}{2}$. There are $\binom{4}{2}$ ways to permute columns in each collection. We thus obtain

$$M_4(D_1) = 4!B_4(D_1) + 4!B_3(D_1) + \left(m_1 + \binom{m_1}{2}\binom{4}{2}\right)$$
$$= 4!B_4(D_1) + 4!B_3(D_1) + \left(m_1 + 6\binom{m_1}{2}\right).$$

Using (2.27) and (2.28), $B_4(D)$ has the following expression:

$$B_4(D) = 4!B_4(D_1)B_4(D_2) + 4!B_3(D_1)B_4(D_2) + \left(m_1 + 6\binom{m_1}{2}\right)B_4(D_2)$$
$$+ \left(m_2 + 6\binom{m_2}{2}\right)(B_4(D_1) + B_3(D_1)) + \binom{m_1}{2}\binom{m_2}{2}. \tag{2.37}$$

For $k = 5$, it is obvious that

$$M_5(D_1) = C_0^{*(1)}B_5(D_1) + C_1^{*(1)}B_4(D_1) + C_2^{*(1)}B_3(D_1) + C^*.$$

Clearly, $C_0^{*(1)} = C_1^{*(1)} = 5!$. The term $C_2^{*(1)}B_3(D_1)$ is identical to the second term $60(m_1 - 2)B_3(D_1)$ in (2.30). The interpretation of the coefficient of $B_3(D_1)$ was also given in Section 2.3. For the constant term, we let h be the number of I_{n_1}'s. Since there are five columns in a collection, h is odd, clearly $h = 1, 3, 5$. For $h = 1$ there are two cases, (a) the remaining four columns are identical, (b) not all of them are the same. Case (a) gives $(m_1 - 1)$ ways to choose a repeating column and $\binom{5}{4}$ ways to permute those five columns in each collection. Case (b) shows that $J_5^2(a_{i_1}, \ldots, a_{i_5}) > 0$, only when two distinct columns occur exact twice. There are $\binom{m_1-1}{2}$ ways to choose two distinct columns and $\binom{5}{2}\binom{3}{2}$ ways to permute those five columns. For $h = 3$, in order to obtain $J_5^2(a_{i_1}, \ldots, a_{i_5}) > 0$, the left two columns in the collection must be the same. The number of ways to choose a column is $\binom{m_1-1}{1}$ and there are $\binom{5}{2}$ ways to permute those five columns. It is obvious that only $J_5^2(I_{n_1}, \ldots, I_{n_1}) > 0$ for $h = 5$. $M_5(D_1)$ can then be obtained

$$\begin{aligned}
M_5(D_1) &= 5!B_5(D_1) + 5!B_4(D_1) + 60(m_1-2)B_3(D_1) \\
&\quad + \left(\binom{m_1-1}{1}\binom{5}{4} + \binom{m_1-1}{2}\binom{5}{2}\binom{3}{2} + \binom{m_1-1}{1}\binom{5}{2} + 1\right) \\
&= 5!B_5(D_1) + 5!B_4(D_1) + 60(m_1-2)B_3(D_1) + 30\binom{m_1-1}{2} + 15m_1 - 14
\end{aligned}$$

Combining the above with (2.31) and (2.32), we get that

$$\begin{aligned}
B_5(D) &= 5!B_5(D_1)B_5(D_2) + 5!B_4(D_1)B_5(D_2) \\
&\quad + 60(m_2-2)\left(B_5(D_1) + B_4(D_1)\right)B_3(D_2) \\
&\quad + 60(m_1-2)B_3(D_1)B_5(D_2) + \left(30\binom{m_1-1}{2} + 15m_1 - 14\right)B_5(D_2) \\
&\quad + (27m_1m_2 - 60m_1 - 60m_2 + 126)B_3(D_1)B_3(D_2) \\
&\quad + (6m_1^2m_2 - 15m_1^2 - 14m_1m_2 + 33m_1 + 8m_2 - 18)B_3(D_2). \qquad (2.38)
\end{aligned}$$

These three quantities $B_3(D)$, $B_4(D)$ and $B_5(D)$ play an important role in the next chapter.

Chapter 3

Applications

3.1 Generals

Section 2.1 presents a simple approach to constructing a large design using two small designs. But, it may be computationally difficult or impractical to assess the goodness of this large design using minimum G_2-aberration by directly evaluating values of the associated B_k for $1 \leq k \leq m$. The properties derived from Sections 2.2-2.4 provide us with a simple way to achieve this.

Theorem 2.2.1 shows that $B_k(D)$ of the doubled design D is a linear combination of $B_k(D_2)$, $B_{k-2}(D_2), \ldots$ of the original design D_2, with a positive coefficient for the leading term $B_k(D_2)$. Theorem 2.3.1 says that $B_k(D)$ of design D is a linear combination of $B_k(D_1)B_k(D_2)$, $B_k(D_1)$ $B_{k-2}(D_2), \ldots$ of the original designs D_1 and D_2, with a positive coefficient for the leading term $B_k(D_1)B_k(D_2)$. Resembling Theorem 2.3.1, Theorem 2.4.1 states that $B_k(D)$ is a linear combination of $B_k(D_1)B_k(D_2)$, $B_k(D_1)B_{k-2}(D_2), \ldots$ of the original designs D_1 and D_2, with a positive coefficient for the leading term $B_k(D_1)B_k(D_2)$. These results imply that if we choose D_1 and D_2 with small B_k's, the corresponding D also has small $B_k(D)$'s. A design with small B_k's can then be obtained using two small minimum G_2-aberration designs.

Specific expressions for the $B_3(D)$ value in Theorems 2.2.1, 2.3.1, and 2.4.1 are given in (2.8), (2.25), and (2.36), respectively. The corresponding $B_4(D)$'s are presented in (2.9), (2.29), and (2.37). We note that in each case, $B_3(D)$ depends on $B_3(D_2)$. However, each $B_4(D)$ contains a constant term. If we choose a design D_2 with $B_3(D_2) = 0$, then we obtain the corresponding D with $B_3(D) = 0$, which means that our approach can construct a large design with $4 \leq R(D) < 5$ using a smaller design D_2 with $R(D_2) \geq 4$, where $R(D)$ and $R(D_2)$ are the generalized resolutions of D and D_2, respectively. We note that $R(D_2) \geq 4$ requires some restriction on the number of factors and the run size. It is well known that design D_2 with $R(D_2) \geq 4$ exists, only if $m_2 \leq n_2/2$. This implies that given a run size n, design D must have $4 \leq R(D) < 5$, if the number of factors is small. On the other hand, D has $3 \leq R(D) < 4$, if the number of factors is large. In most cases, $B_3(D)$, $B_4(D)$, and $B_5(D)$ are enough to compare D's for a given m and n.

In the next two sections, we use regular designs with $7 \le m \le 63$ factors and 64 runs to illustrate our approach.

3.2 Regular 64-Run Designs with $7 \le m \le 63$ Factors

In order to evaluate our approach, the best design from our approach must be found for each m value. According to the value of m, we consider two situations, *(i) m is a composite number; and (ii) m is a prime.*

Designs under situation (i) can be constructed by our approach directly. For each composite number m, we consider all possible combinations of D_1 and D_2 that satisfy $m_1 m_2 = m$ and $n_1 n_2 = 64$. For each combination, we use the corresponding minimum aberration designs D_1 and D_2. Based on the type of D_1, three components $(A_3(D), A_4(D), A_5(D))$ of the word length pattern can be computed using the associated results in Sections 2.2-2.4. We then compare all the resulting designs by the minimum aberration criterion, and the design with the least aberration is the best design from our approach. This best design is then compared with the corresponding minimum aberration design in Section 3.3.

For instance, for $m = 24$, all the possible small designs, D_1 and D_2 that satisfy $m_1 m_2 = 24$ and $n_1 n_2 = 64$, are given in Table 3.1, where $D(i)$ is the saturated design of i independent factors with a column of 1's added. This table also contains A_3, A_4, A_5 values of the minimum aberration designs D_1 and D_2 in each combination, respectively. The corresponding (A_3, A_4, A_5) of design D is computed and given in the last column of Table 3.1. Obviously, design D with $(A_3(D), A_4(D), A_5(D)) = (0, 370, 0)$ has the least aberration, and is the best design for $m = 24$ from our approach.

Table 3.1: All possible D_1's and D_2's for designs 2^{24-18}

Number of Factors		Runs		D	D_1	D_2	D
m_1	m_2	n_1	n_2	$D_1 \otimes D_2$	(A_3, A_4, A_5)	(A_3, A_4, A_5)	(A_3, A_4, A_5)
2	12	4	16	$2^2 \otimes 2^{12-8}$	$(0,0,0)$	$(16,39,48)$	$(0,378,0)$
		2	32	$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \otimes 2^{12-7}$	$(0,0,0)$	$(0,38,0)$	$(0,370,0)$
4	6	8	8	$2^{4-1} \otimes 2^{6-3}$	$(0,1,0)$	$(4,3,0)$	$(0,378,0)$
		4	16	$D(2) \otimes 2^{6-2}$	$(1,0,0)$	$(0,30,0)$	$(0,378,0)$
8	3	8	8	$D(3) \otimes 2^3$	$(7,7,0)$	$(0,0,0)$	$(0,378,0)$
		16	4	$2^{8-4} \otimes 2^{3-1}$	$(0,14,0)$	$(1,0,0)$	$(0,378,0)$

For prime values of m, our approach does not apply directly. In this case, a design with a prime number m of factors can be obtained by the method of deleting one column from a design with $m+1$ factors, which can be constructed by our approach. It can be done as follows.

Each time we delete one column from the best design constructed with $m + 1$ factors, we then compute A_3, A_4, A_5 values of the design containing the remaining columns. Obviously, the total number of designs with m factors is $m + 1$. The best design with m factors having the least aberration is then found by comparing all those $m + 1$ designs.

Table 3.2: (A_3, A_4, A_5)'s of the best designs from our approach with $7 \leq m \leq 63$ factors

Number of Factors	(A_3, A_4, A_5)	Number of Factors	(A_3, A_4, A_5)
7*	$(0, 3, 0)$	36	$(64, 1337, 4544)$
8	$(0, 6, 0)$	37*	$(80, 1401, 5760)$
9	$(0, 9, 0)$	38	$(96, 1483, 7040)$
10	$(0, 10, 0)$	39*	$(112, 1579, 8400)$
11*	$(0, 10, 0)$	40	$(128, 1694, 9856)$
12	$(0, 15, 0)$	41*	$(144, 1822, 11432)$
13*	$(0, 19, 24)$	42	$(160, 1970, 13136)$
14	$(0, 29, 32)$	43*	$(176, 2146, 14960)$
15	$(0, 30, 60)$	44	$(192, 2335, 16960)$
16	$(0, 52, 64)$	45	$(210, 2520, 19215)$
17*	$(0, 64, 96)$	46	$(224, 2773, 21504)$
18	$(0, 84, 128)$	47*	$(240, 3025, 24080)$
19*	$(0, 100, 192)$	48	$(256, 3300, 26880)$
20	$(0, 125, 256)$	49	$(294, 3479, 29841)$
21	$(0, 210, 0)$	50	$(304, 3836, 33184)$
22	$(0, 255, 0)$	51*	$(328, 4140, 36744)$
23*	$(0, 307, 0)$	52	$(352, 4469, 40608)$
24	$(0, 370, 0)$	53*	$(376, 4821, 44800)$
25*	$(0, 438, 0)$	54	$(400, 5199, 49344)$
26	$(0, 518, 0)$	55*	$(424, 5603, 54264)$
27*	$(0, 606, 0)$	56	$(448, 6034, 59584)$
28	$(0, 707, 0)$	57*	$(476, 6482, 65240)$
29*	$(0, 819, 0)$	58	$(504, 6958, 71344)$
30	$(0, 945, 0)$	59*	$(532, 7462, 77924)$
31*	$(0, 1085, 0)$	60	$(560, 7995, 85008)$
32	$(0, 1240, 0)$	61*	$(590, 8555, 92568)$
33*	$(16, 1240, 1120)$	62	$(620, 9145, 100688)$
34	$(32, 1256, 2240)$	63	$(651, 9765, 109368)$
35*	$(48, 1288, 3376)$		

This delete-one-column method is in fact quite versatile. It can also be applied when m is a composite number, and can sometimes produce better designs than the direct method. Table 3.2 provides three components $(A_3(D), A_4(D), A_5(D))$ of the word length pattern of the best design for each $7 \leq m \leq 63$ found by our method. The entries with a "$*$" identify those cases where the designs are found using the delete-one-method. Noteworthy are the entries for $m = 25, 27, 33, 35, 39, 51, 55,$ and 57. Although these m values are composite, the best designs given in the table are actually obtained using the method of deleting one column.

3.3 Comparison with Minimum Aberration Designs

To assess the goodness of the best designs constructed from our approach, we compare them with the corresponding minimum aberration designs. We separately consider two ranges of m values, $7 \leq m \leq 32$ and $33 \leq m \leq 63$. The complete catalogue for regular 64-run designs with $7 \leq m \leq 32$ factors of resolution 4 is available in Chen, Sun, and Wu (1993) and Xu (2009). For designs with $33 \leq m \leq 63$ factors, there is no summary available for the corresponding minimum aberration designs. Hence, we compute A_3, A_4, A_5 values for the minimum aberration designs with $33 \leq m \leq 63$ factors, and then make a comparison.

The complete catalogue for $7 \leq m \leq 32$ includes three components (A_3, A_4, A_5) of the word length pattern for each nonisomorphic design with m factors, which is given in Table 3.3. The (A_3, A_4, A_5) values of the best designs from our approach are also given in Table 3.3. From this table, we see that (A_3, A_4, A_5)'s of designs with $m = 15, 19, 20$ from our approach are identical to those of the corresponding minimum aberration designs. For $m = 30, 31$, and 32, the designs constructed by our approach are equivalent to the minimum aberration designs, which is because these designs are unique. Generally, as the m value increases, the result of (A_3, A_4, A_5) derived from our approach is getting closer to that from the minimum aberration design.

Table 3.3: (A_3, A_4, A_5)'s of minimum aberration (MA) designs and those of the corresponding best designs from our approach for $7 \leq m \leq 32$

Number of Factors (m)	MA designs (A_3, A_4, A_5)	Constructed designs (A_3, A_4, A_5)	Number of Factors (m)	MA designs (A_3, A_4, A_5)	Constructed designs (A_3, A_4, A_5)
7	$(0, 0, 0)$	$(0, 3, 0)$	20	$(0, 125, 256)$	$(0, 125, 256)$
8	$(0, 0, 2)$	$(0, 6, 0)$	21	$(0, 204, 0)$	$(0, 210, 0)$
9	$(0, 1, 4)$	$(0, 9, 0)$	22	$(0, 250, 0)$	$(0, 255, 0)$
10	$(0, 2, 8)$	$(0, 10, 0)$	23	$(0, 304, 0)$	$(0, 307, 0)$
11	$(0, 4, 14)$	$(0, 10, 0)$	24	$(0, 365, 0)$	$(0, 370, 0)$
12	$(0, 6, 24)$	$(0, 15, 0)$	25	$(0, 435, 0)$	$(0, 438, 0)$
13	$(0, 14, 28)$	$(0, 19, 24)$	26	$(0, 515, 0)$	$(0, 518, 0)$
14	$(0, 22, 40)$	$(0, 29, 32)$	27	$(0, 605, 0)$	$(0, 606, 0)$
15	$(0, 30, 60)$	$(0, 30, 60)$	28	$(0, 706, 0)$	$(0, 707, 0)$
16	$(0, 43, 81)$	$(0, 52, 64)$	29	$(0, 819, 0)$	$(0, 819, 0)$
17	$(0, 59, 108)$	$(0, 64, 96)$	30	$(0, 945, 0)$	$(0, 945, 0)$
18	$(0, 78, 144)$	$(0, 84, 128)$	31	$(0, 1085, 0)$	$(0, 1085, 0)$
19	$(0, 100, 192)$	$(0, 100, 192)$	32	$(0, 1240, 0)$	$(0, 1240, 0)$

To make a comparison for designs with $33 \leq m \leq 63$ factors, we firstly deduce (A_3, A_4, A_5)'s of minimum aberration designs using the complementary designs. Let H be the saturated design of 64 runs for 63 factors, and $D \subseteq H$ be a design of m factors. The corresponding complementary design with $\bar{m}$ factors is denoted by $\bar{D} = H \setminus D$, where $\bar{m} = 63 - m$. Tang and Wu (1996) proved that

$$A_3(D) = constant - A_3(\bar{D}), \tag{3.1}$$

$$A_4(D) = constant + A_3(\bar{D}) + A_4(\bar{D}), \tag{3.2}$$

and

$$A_5(D) = constant - (2^{f-1} - m)A_3(\bar{D}) - A_4(\bar{D}) - A_5(\bar{D}), \tag{3.3}$$

where f is the number of independent factors. In our case, $f = 6$. The above equations give us a method of computing $A_3(D), A_4(D)$, and $A_5(D)$ from $A_3(\bar{D}), A_4(\bar{D})$, and $A_5(\bar{D})$. This raises the question of how to find the values of $A_3(\bar{D}), A_4(\bar{D})$, and $A_5(\bar{D})$, if design D has minimum aberration. Tang and Wu (1996) also suggested a rule for identifying the minimum aberration designs.

Rule 3.3.1. *A design D^* has minimum aberration if:*

(i) $A_3(\bar{D}^*) = \max A_3(\bar{D})$ *over all* $\left|\bar{D}\right| = \bar{m}$,

(ii) $A_4(\bar{D}^*) = \min\left\{A_4(\bar{D}) : A_3(\bar{D}) = A_3(\bar{D}^*)\right\}$;

(iii) $A_5(\bar{D}^*) = \max\left\{A_5(\bar{D}) : A_3(\bar{D}) = A_3(\bar{D}^*) \text{ and } A_4(\bar{D}) = A_4(\bar{D}^*)\right\}$ *and*

(iv) $\bar{D}^*$ *is the unique set (up to isomorphism) satisfying (iii).*

It is obvious that, first of all, we should find 64-run design $\bar{D}$ with the maximum value of A_3. To obtain such designs, we use the following fact.

Fact 3.3.2. *For a regular n-run design with $3 \le m \le n/2$ factors, the maximum A_3 is attained only by the designs constructed by repeating a design with m factors and n^* runs, where $n^* = 2^q$ with q satisfying $2^{q-1} \le m \le 2^q - 1$. Clearly, $A_3 = 0$ for $m = 1$ and 2.*

By the above fact, we have the following five results:

i Design $\bar{D}$ contains any one or two columns from H, if $\bar{m} = 1$ or 2;

ii Design $\bar{D}$ is found from 4-run designs, if $\bar{m} = 3$;

iii Design $\bar{D}$ is found from 8-run designs, if $4 \le \bar{m} \le 7$;

iv Design $\bar{D}$ is found from 16-run designs, if $8 \le \bar{m} \le 15$;

v Design $\bar{D}$ is found from 32-run designs, if $16 \le \bar{m} \le 31$.

According to the above results and Rule 3.3.1, we obtain the corresponding unique design $\bar{D}$ of $\bar{m}$ factors for $8 \le \bar{m} \le 31$ from the complete catalogue for 16-run designs and 32-run designs. The 4-run design with 3 factors and 8-run designs with $5 \le \bar{m} \le 7$ factors are unique for each $\bar{m}$. For $\bar{m} = 4$, there are two nonisomorphic designs, and we choose the one with large $A_3(\bar{D})$. The final results are summarized in Table 3.4. Tang and Wu (1996) also gave some recursive formulas, which can be used to determine the constants in (3.1)-(3.3). After some calculations, we obtain

$$A_3(D) = \frac{1}{3}\left[\binom{\bar{m}}{2} + \binom{m}{2} - \frac{m\bar{m}}{2}\right] - A_3(\bar{D}), \tag{3.4}$$

$$A_4(D) = \frac{1}{4}\left[\frac{m-3}{3}\binom{\bar{m}}{2} - \frac{\bar{m}+1}{3}\binom{m}{2} + \frac{m\bar{m}}{3} - \binom{\bar{m}}{3} + \binom{m}{3}\right] + A_3(\bar{D}) + A_4(\bar{D}), \tag{3.5}$$

and

$$\begin{aligned} A_5(D) = \frac{1}{5}\Bigg[&\binom{\bar{m}}{4} + \frac{1}{4}\binom{\bar{m}}{3}(4-m) + \frac{1}{6}\binom{\bar{m}}{2}(3\bar{m} - 7m + 9) + \frac{1}{6}\binom{m}{2}\binom{\bar{m}}{2} + \binom{m}{4} \\ &-\frac{1}{4}\binom{m}{3}(\bar{m}-1) + \frac{1}{12}\binom{m}{2}(3\bar{m} - 4m + 13) + \frac{1}{24}m\bar{m}\,(7m - \bar{m} - 21)\Bigg] \\ &-(2^{f-1} - m)A_3(\bar{D}) - A_4(\bar{D}) - A_5(\bar{D}). \end{aligned} \tag{3.6}$$

Using (3.4)-(3.6), the A_3, A_4, A_5 values of minimum aberration designs with $33 \le m \le 63$ are readily determined, and also contained in Table 3.4.

Table 3.5 provides (A_3, A_4, A_5)'s of minimum aberration designs and those of best designs from our approach. The results are very satisfying. About two-thirds of our best designs give the same (A_3, A_4, A_5) as the corresponding minimum aberration designs. Even though some cases give different results, the discrepancies are rather small. Among those cases with different results, there are 9 cases giving the identical A_3 values. Only 2 cases have different A_3 values, but the differences are quite small.

Table 3.4: (A_3, A_4, A_5) of design $\bar{D}$ and that of the corresponding MA design D

$\bar{D}$		D	
Number of Factors ($\bar{m}$)	(A_3, A_4, A_5)	Number of Factors (m)	(A_3, A_4, A_5)
30	$(140, 945, 4368)$	33	$(16, 1240, 1120)$
29	$(126, 819, 3640)$	34	$(32, 1256, 2240)$
28	$(113, 706, 3012)$	35	$(48, 1288, 3376)$
27	$(101, 605, 2473)$	36	$(64, 1336, 4544)$
26	$(90, 515, 2013)$	37	$(80, 1400, 5760)$
25	$(80, 435, 1623)$	38	$(96, 1480, 7040)$
24	$(71, 365, 1292)$	39	$(112, 1577, 8402)$
23	$(63, 304, 1015)$	40	$(128, 1691, 9860)$
22	$(56, 251, 784)$	41	$(144, 1822, 11432)$
21	$(50, 205, 592)$	42	$(160, 1970, 13136)$
20	$(45, 175, 453)$	43	$(176, 2145, 14960)$
19	$(41, 147, 337)$	44	$(192, 2334, 16960)$
18	$(38, 126, 252)$	45	$(208, 2543, 19136)$
17	$(36, 112, 196)$	46	$(224, 2773, 21504)$
16	$(35, 105, 168)$	47	$(240, 3025, 24080)$
15	$(35, 105, 168)$	48	$(256, 3300, 26880)$
14	$(28, 77, 112)$	49	$(280, 3556, 29904)$
13	$(22, 55, 72)$	50	$(304, 3836, 33184)$
12	$(17, 38, 44)$	51	$(328, 4140, 36744)$
11	$(13, 25, 25)$	52	$(352, 4468, 40608)$
10	$(10, 15, 12)$	53	$(376, 4820, 44801)$
9	$(8, 10, 4)$	54	$(400, 5199, 49344)$
8	$(7, 7, 0)$	55	$(424, 5603, 54264)$
7	$(7, 7, 0)$	56	$(448, 6034, 59584)$
6	$(4, 3, 0)$	57	$(476, 6482, 65240)$
5	$(2, 1, 0)$	58	$(504, 6958, 71344)$
4	$(1, 0, 0)$	59	$(532, 7462, 77924)$
3	$(1, 0, 0)$	60	$(560, 7995, 85008)$
2	$(0, 0, 0)$	61	$(590, 8555, 92568)$
1	$(0, 0, 0)$	62	$(620, 9145, 100688)$
0	$(0, 0, 0)$	63	$(651, 9765, 109368)$

Table 3.5: (A_3, A_4, A_5)'s of MA designs and those of the corresponding best designs from our approach for $33 \le m \le 63$

Number of Factors (m)	MA designs (A_3, A_4, A_5)	Constructed designs (A_3, A_4, A_5)	Number of Factors (m)	MA designs (A_3, A_4, A_5)	Constructed designs (A_3, A_4, A_5)
33	$(16, 1240, 1120)$	$(16, 1240, 1120)$	49	$(280, 3556, 29904)$	$(294, 3479, 29841)$
34	$(32, 1256, 2240)$	$(32, 1256, 2240)$	50	$(304, 3836, 33184)$	$(304, 3836, 33184)$
35	$(48, 1288, 3376)$	$(48, 1288, 3376)$	51	$(328, 4140, 36744)$	$(328, 4140, 36744)$
36	$(64, 1336, 4544)$	$(64, 1337, 4544)$	52	$(352, 4468, 40608)$	$(352, 4469, 40608)$
37	$(80, 1400, 5760)$	$(80, 1401, 5760)$	53	$(376, 4820, 44801)$	$(376, 4821, 44800)$
38	$(96, 1480, 7040)$	$(96, 1483, 7040)$	54	$(400, 5199, 49344)$	$(400, 5199, 49344)$
39	$(112, 1577, 8402)$	$(112, 1579, 8400)$	55	$(424, 5603, 54264)$	$(424, 5603, 54264)$
40	$(128, 1691, 9860)$	$(128, 1694, 9856)$	56	$(448, 6034, 59584)$	$(448, 6034, 59584)$
41	$(144, 1822, 11432)$	$(144, 1822, 11432)$	57	$(476, 6482, 65240)$	$(476, 6482, 65240)$
42	$(160, 1970, 13136)$	$(160, 1970, 13136)$	58	$(504, 6958, 71344)$	$(504, 6958, 71344)$
43	$(176, 2145, 14960)$	$(176, 2146, 14960)$	59	$(532, 7462, 77924)$	$(532, 7462, 77924)$
44	$(192, 2334, 16960)$	$(192, 2335, 16960)$	60	$(560, 7995, 85008)$	$(560, 7995, 85008)$
45	$(208, 2543, 19136)$	$(210, 2520, 19215)$	61	$(590, 8555, 92568)$	$(590, 8555, 92568)$
46	$(224, 2773, 21504)$	$(224, 2773, 21504)$	62	$(620, 9145, 100688)$	$(620, 9145, 100688)$
47	$(240, 3025, 24080)$	$(240, 3025, 24080)$	63	$(651, 9765, 109368)$	$(651, 9765, 109368)$
48	$(256, 3300, 26880)$	$(256, 3300, 26880)$			

Chapter 4

Concluding Remarks

Starting with any two fractional factorials, either regular or nonregular, D_1 with m_1 factors and n_1 runs and D_2 with m_2 factors and n_2 runs, we present a general approach to constructing a large design D with $m = m_1m_2$ factors and $n = n_1n_2$ runs. For any k with $1 \leq k \leq m_1m_2$, based on the type of D_1 we derive three equations that connect $B_k(D)$ to $B_s(D_1)$'s and $B_t(D_2)$'s for $s, t \leq k$. These results imply that the best design D from our approach can be constructed by choosing two minimum G_2-aberration designs D_1 and D_2. Regular 64-run designs of $7 \leq m \leq 63$ factors are used to evaluate this approach. The findings are very promising - the A_3, A_4, A_5 values of the best designs from our approach are the same as or very close to those of the corresponding minimum aberration designs.

In this project, the evaluation of our approach focuses on designs of 64 runs. One future work would be to evaluate our approach by looking at designs with 128 runs, as all minimum aberration designs of 128 runs are known, which can be used to compare with designs constructed from our approach. On the other hand, Deng and Tang (1999) defined another generalization of minimum aberration criterion, minimum G-aberration, to compare factorials for a given number of factors and run size. We may then use the general construction to obtain minimum G-aberration designs, which would be another interesting topic for the future work.

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