

MODELING MORTALITY RATES WITH THE LINEAR LOGARITHM HAZARD TRANSFORM APPROACHES

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Abstract

In this project, two approaches based on the linear logarithm hazard transform (LLHT) to modeling mortality rates are proposed. Empirical observations show that there is a linear relationship between two sequences of the logarithm of the forces of mortality (hazard rates of the future lifetime) for two years. The estimated two parameters of the linear relationship can be used for forecasting mortality rates. Deterministic and stochastic mortality rates with the LLHT, Lee-Carter and CBD models are predicted, and their corresponding forecasted errors are calculated for comparing the forecasting performances. Finally, applications to pricing some mortality-linked securities based on the forecasted mortality rates are presented for illustration.

keywords: linear logarithm hazard transform; vector autoregressive model; confidence interval; mortality-linked security; q-forward; credit tranche technique

To my parents!

“Don’t worry, Gromit. Everything’s under control!”
— *The Wrong Trousers*, AARDMAN ANIMATIONS, 1993

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Chapter 1

Introduction

1.1 The motivation of this project

It is very crucial for life insurance companies and annuity providers to use proper projected mortality rates in their pricing calculations. If the annuity providers underestimate longevity risk, they will suffer from financial stress. If mortality rates increase, the life insurers need to pay the death benefits earlier than expected.

Meanwhile, life insurance companies and annuity providers can take some steps to hedge their mortality risks. Blake *et al.* (2006) systematically summarized the ways of hedging mortality risk as follows:

1. taking on mortality risk as a legitimate business risk,
2. transferring mortality risk to reinsurers,
3. diversifying mortality risk by engaging in business in different regions or across different products,
4. internally hedging the mortality risk by balancing the portions of the annuity and insurance policies,
5. promoting more participating business to transfer mortality risk to the policyholders, and
6. using mortality-linked securities.

The first and second items are the common business approaches to mitigating life contingency risks with the costs of higher premium and reinsurance, respectively. Some of these items are involving changing business structures (e.g., the third item), and the fourth and the fifth items could be extremely costly and impractical in some sense. It has not been long for the academic and financial institutions to study and design the mortality-linked securities. This project is interested in taking a look into one of the mortality-linked securities in the market. Pricing these mortality securities relies on a sound projection of future mortality rates.

Hence, the main topic of this project is about mortality projection methods. Cairns *et al.* (2008) discussed the standard of judging mortality projection models. They mentioned that a sound mortality projection method needs to be able to capture the trend of mortality improvement or deterioration for different ages in a certain population. They also stated that predicting the uncertainty of the projected mortality rates (or the confidence interval of the projected mortality rates) is also an important consideration in choosing a proper mortality projection method.

1.2 Outline

This project is organized as follows. Chapter 2 reviews a variety of mortality projection methods, discusses their mechanism, compares and contrasts their characteristics, and explores their variations. These mortality projection models include the Lee-Carter (LC) model, the Cairns-Blake-Dowd (CBD) model, the linear hazard transform (LHT) model and the linear log-hazard transform (LLHT) model.

Chapter 3 presents the mathematical forms of some mortality projection models given in Chapter 2, including two projection approaches for the LLHT model. Parameter estimation approaches for each of these mortality projection method are presented. Two new variations of the LLHT models are then introduced in Chapter 4.

q-Forward is one of the mortality linked securities studied in the project. Chapter 5 discusses the mechanism of q-forwards in detail. The content includes descriptions of the duties of the various counterparties in the transaction and the pricing of the product. In addition, this chapter describes a new mortality-linked security which is designed based on characteristics of the q-forward and the tranche idea.

In Chapter 6, empirical mortality data from three countries are used in a numerical comparisons of the mortality projection models presented in Chapter 3 and 4. The countries selected for this comparison are Japan, United Kingdom and United States of America. All three of these countries are well-developed with large populations and their own demographic characteristics. Japan has been famous for the longevity of its population. The USA is a developed country with leading medical science, together with a serious obesity problem. The age range from 25 to 84 is chosen for this project.

Chapter 2

Literature review

There is an extensive list of mortality projection models. In the first section of this chapter, we review several influential ones. In the second section of this chapter, we review several mortality-linked securities.

2.1 Mortality projection models

As summarized by Booth and Tickle (2008), mortality projection models typically involve three variables: age, period (time) and cohort. Accordingly, mortality forecasting models can be classified as one-factor, two-factor, or multi-factor models. A one-factor model assumes that the mortality rate is a function of age, and that the age pattern is stable over the period in question. A two-factor model usually takes the age and period variables or sometimes the age and cohort variables, as the factors for the mortality rate function. Multi-factor models, including generalized linear models (GLM), have been applied to a range of mortality forecasting modeling issues.

The Lee-Carter mortality model has long been a popular mortality projection method especially for long term projections since its introduction by Lee and Carter (1992). The Lee-Carter model is a two-factor model, extracting two age-specific elements for every age and a time-varying effect for every fitted time from the central death rates. In the projection phrase, the model keeps the age-specific element of each age and projects the future time-varying index by a time series model that follows a random walk with drift. The derivatives of the Lee-Carter model fall into two categories: some focus on improving

the modeling of the time-varying element, while others try to adapt the Lee-Carter model to a GLM framework.

Modifications of the time-varying element in the Lee-Carter model include a number of approaches. When using a random walk to fit the time-varying element, Booth *et al.* (2002) found that the drift changes from time to time. They then tried to estimate the optimal period for the drift to be constant. By contrast, De Jong and Tickle (2006) used the latent random process approach to solve the issue that the drift in the random walk model is not constant over time. The autoregressive integrated moving average (ARIMA) method for predicting the time-varying index was proposed by CMI (2007). Deng *et al.* (2012) introduced a stochastic diffusion model with a double-exponential jump diffusion process in the hope of further increasing the accuracy of the projected mortality rates, as that is the key to the pricing of mortality-linked securities.

Other models experimented with increasing the precision of the projected mortality rates by adding more components to the original Lee-Carter model, effectively making it a GLM. For example, Renshaw and Haberman (2003) proposed a variation of the Lee-Carter model with an additional dependent period effect, and later Renshaw and Haberman (2006) generalized the Lee-Carter model by including a cohort effect.

As an alternative to the Lee-Carter-based models, a two-factor stochastic mortality model was introduced by Cairns, Blake and Dowd (2006) (abbreviated as the CBD model) with specific emphasis on modeling the post-age-60 portion of the mortality curve. In the CBD model, one factor drives the evolution of mortality rates at all ages while the other factor has more influence on the older ages than the younger ones. In addition, the CBD model is constructed upon the logit of mortality rates. The logit of a number q between 0 and 1, $\text{logit}(q) = \log\left(\frac{q}{1-q}\right)$, can broaden the scale of the original number. Derivatives of the CBD model can be found by adding effects. Cairns *et al.* (2007) included an additional age-period effect, quadratic in age, and a cohort effect, a function of the approximate year of birth, to the basic CBD model. Later Cairns *et al.* (2008) defined a more complicated cohort effect element.

Noting from empirical mortality data that there is a linear relationship between two sequences of the force of mortality from two different risk classes, Tsai and Jiang (2010) introduced the linear hazard transform (LHT) model. The two risk classes can be two genders,

two calendar years, or two countries, etc. Furthermore, in order to prevent the modeled mortality rates from falling below zero, Tsai (2012) revised the linear hazard transform model to the linear logarithm hazard transform (LLHT) model by replacing the mortality hazard rate with the natural logarithm of the mortality hazard rate in the original linear model.

For the LHT model, Tsai and Jiang (2010) proposed the arithmetic and geometric growth methods for forecasting mortality rates, which can also be applied to the LLHT model. They were named as arithmetic and geometric for the way they project the future values of the two parameters of the LHT (or LLHT) model. The advantages of the arithmetic and geometric growth methods for the LHT and LLHT models over the Lee-Carter and CBD models can be summarized as follows. First, the arithmetic and geometric growth methods require mortality rates for only two years, whereas the Lee-Carter and CBD models need mortality rates for more than two years. This feature is very important in cases where a mortality table is not produced very often. For example, there are only two official Commissioners Standard Ordinary (CSO) mortality tables (1980 CSO and 2001 CSO) published by the Society of Actuaries. Second, the LHT and LLHT models are essentially linear regression models, which means that the estimates of the parameters are effortless to obtain. Hence these mortality projection approaches are fairly easy to comprehend.

2.2 Mortality-linked security models

Sophisticated methods of pricing mortality-linked securities have been studied. We summarize several famous pricing methods including the Wang transform introduced by Wang (2000), the canonical method of Li and Ng (2011), the risk-neutral dynamics of death/survival rates proposed by Cairns *et al.* (2006), and the instantaneous Sharpe ratio method of Milevsky *et al.* (2005).

One of the most notable contributions to the pricing of mortality-linked securities comes from a transform proposed by Wang (2000). The Wang transform is a distortion operator which creates a margin for the risk premium. The range of applications for the Wang transform is not restricted to mortality-linked bonds but reaches to any asset and liability. An example of parameter estimation in the Wang transform formula can be found in Wang (2004). The Wang transform is the basis for many mortality-linked securities, and was first

adopted by Lin and Cox (2005) to price a variety of mortality-linked bonds and mortality-linked swaps.

In addition to introducing CBD stochastic mortality projection model, Cairns *et al.* (2006) also proposed and developed a corresponding pricing method for mortality-linked bonds. It is a market risk-adjusted pricing method that makes allowances for both the underlying stochastic mortality rates and the parameter risk. However, this pricing method is solely dependent on the CBD stochastic mortality projection model. More specifically, the authors assumed that there are dynamics under a risk-adjusted pricing measure Q (the risk-adjusted measure or equivalent-martingale measure) which are equivalent to the current real world measure P in the sense of probability. The risk-adjusted measure Q is used to re-estimate the time-varying parameter in the CBD model to reflect the market price. The pricing of EIB/BNP longevity bond was used as an example. By equating the initial value of the bond to the bond value estimated by the risk-adjusted pricing method, the parameter of the risk-adjusted pricing model can be estimated.

Li and Ng (2011) introduced a canonical valuation method for mortality-linked securities. The idea of the canonical valuation is originated from valuing financial derivatives. Stutzer (1996) stated that “canonical valuation uses historical time series to predict the probability distribution of the discounted value of primary assets’ discounted prices plus accumulated dividends at any future date.” This method uses non-parametric estimation, and thus it does not necessarily require a stochastic mortality model. Unlike the Wang transform, it does not require the market price to predict the prices of the securities and the risk-neutral dynamics of the death/survival probabilities, which is very valuable in the case of a handful of mortality-linked securities in the market so far. The intuitive meaning of the canonical valuation can be understood from the perspective of mathematics, geography and economics. In the economic sense, the principle of canonical valuation is related to expected utility hypothesis.

Milevsky *et al.* (2005) presented another mortality risk pricing method by awarding the mortality risk holder a risk premium which is the product of the Sharpe ratio and the standard deviation of the portfolio. Coughlan *et al.* (2007) argued that Milevsky’s method can be used to model the fixed mortality rates in a q -forward. Milevsky’s method is a simple transform for the projected mortality rates and is used to price q -forwards in this project.

There are other techniques not relating to the mortality rate transform that are involved in the pricing of mortality-linked securities. Most of them are techniques from the long-researched area of financial securities. Credit tranche techniques identify the “risk-yield” relationship and thus serve as a practical and easy way to design a risk-related product. The credit tranche technique was first applied to modeling mortality-linked bonds by Liao *et al* (2007) for transferring longevity risks to the capital market. Kim and Choi (2011) modeled inverse survivor bonds using the tranche percentile method, and concluded that the mortality-linked inverse survivor bonds have a low-cost advantage over the traditional bonds with respect to the risk of losing some or all of future coupons when more people are alive than expected. Credit tranche techniques can be used to avoid the catastrophic mortality risk for the issuer of mortality-linked securities by setting a maximum tranche percentile.

Chapter 3

The models and assumptions

3.1 Actuarial mathematics preliminaries

In this chapter, some well-known mortality models are introduced and a new one is proposed; the associated parameters are estimated. To begin, some actuarial concepts need to be explained. More details of these actuarial concepts can be found in Bowers *et al.* (1997).

Let the random variable $T(x)$ represent the future lifetime of an individual aged x . The distribution function of $T(x)$ is denoted by

$${}_tq_x \triangleq F_{T(x)}(t) = Pr\{T(x) \leq t\} = 1 - {}_tp_x,$$

where

$${}_tp_x \triangleq S_{T(x)}(t) = Pr\{T(x) > t\}$$

is the survival function of $T(x)$. The hazard rate of the random variable $T(x)$, $\mu_x(t)$, is defined as

$$\mu_x(t) = \frac{f_{T(x)}(t)}{S_{T(x)}(t)} = -\frac{d}{dt} \ln S_{T(x)}(t),$$

where $f_{T(x)}(t)$ is the density function of $T(x)$. Thus the survival function can be expressed in terms of the mortality hazard rate as

$${}_tp_x = e^{-\int_0^t \mu_x(s) ds}.$$

This symbol, $\mu_x(s)$, is widely known as the force of mortality in demography. Under the assumption that the force of mortality is constant between two consecutive integer ages, the

one-year survival probability of an individual aged x can be expressed as

$$p_x = e^{-\mu_x},$$

where μ_x is the constant force of mortality of an individual aged x . In actuarial science, we commonly use S for the survival function of an individual at birth, that is, $S = S_{T(0)}$. Since $T(x)$ is the future lifetime of an individual aged x , the survival function of $T(x)$ can be written as the survival probability of the individual aged $x + t$ conditioning on the survival probability of the individual aged x , that is,

$${}_t p_x = \frac{S(x+t)}{S(x)}.$$

A discrete n -year term life insurance pays a death benefit of \$1 at the end of the year of death of the insured, if the insured dies within n years of issue. The net single premium (NSP) at issue of this policy for an individual aged x , denoted as $A_{x:\overline{n}|}^1$, is given by

$$A_{x:\overline{n}|}^1 = \sum_{k=1}^n {}_{k-1|}q_x \cdot v^k = \sum_{k=1}^n {}_{k-1}p_x \cdot q_{x+k-1} \cdot v^k,$$

where $v = 1/(1+i)$ is the discount factor and i is the annual effective rate of interest.

An n -year temporary annuity-due is an annuity with a payment of \$1 made at the beginning of each year as long as the annuitant survives, with up to n payments. Its NSP, denoted as $\ddot{a}_{x:\overline{n}|}$, is given by

$$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} {}_k p_x \cdot v^k.$$

If the annuity payment is made at the end of each year as long as the annuitant survives limited to a maximum of n payments, it is called an n -year temporary life annuity-immediate and its NSP, denoted by $a_{x:\overline{n}|}$, is given as

$$a_{x:\overline{n}|} = \sum_{k=1}^n {}_k p_x \cdot v^k.$$

3.2 The Lee-Carter and CBD mortality models

3.2.1 Lee-Carter model

The Lee-Carter model uses the natural logarithm of the central death rates to measure the age effect and time effect. The central death rate at age x in year t , $m_{x,t}$, is defined

as $D_{x,t}$, the number of deaths aged x last birthday at the date of death during year t divided by $E_{x,t}$, the average population aged x last birthday during year t . Commonly, there are two approximations to the central death rate $m_{x,t}$ with the mortality rate of an individual aged x at time t , $q_{x,t}$. The first approach, $q_{x,t} = 1 - \exp(-m_{x,t})$, is based on the assumption of constant force of mortality within each integer age. The second approach, $q_{x,t} = m_{x,t}/(1 + 0.5m_{x,t})$, is under the assumption of uniform distribution of deaths (UDD) within each integer age. In this project, the former approximation is adopted for the data transformation between $m_{x,t}$ and $q_{x,t}$. The Lee-Carter model is given by

$$\ln(m_{x,t}) = a_x + b_x k_t + \epsilon_{x,t},$$

where

- both a_x and b_x are age-specific constants,
- k_t is the time-varying index,
- $\epsilon_{x,t}$ is the error term and is assumed to follow a normal distribution with mean zero and to be independent of age x and time t , and
- the mortality data from time t_L to t_U and from age x_L to x_U for age are used for fitting.

As Lee and Carter (1992) explained, under their model e^{a_x} can depict the general shape of the central death rate for age x across the projection period, while b_x can reveal the relative changes in response to t since $d\ln(m_{x,t})/dt = b_x dk_t/dt$. The parameter estimation for the Lee-Carter model is not unique, and is subject to two constraints, $\sum_t k_t = 0$ and $\sum_x b_x = 1$. The first constraint is a natural constraint, which leads a_x to be the average of $\ln(m_{x,t})$ over time t . That is,

$$\hat{a}_x = \frac{1}{t_U - t_L + 1} \sum_{t=t_L}^{t_U} \ln(m_{x,t}).$$

The original paper suggested using the singular value decomposition (SVD) method to find $\{b_x\}$ and $\{k_t\}$ which minimize the sum of least squared errors. Alternatively, the second constraint gives the estimation of k_t by

$$\hat{k}_t = \sum_{x=x_L}^{x_U} [\ln(m_{x,t}) - \hat{a}_x]$$

for each year t , and \hat{b}_x can be obtained by regressing $[\ln(m_{x,t}) - \hat{a}_x]$ on \hat{k}_t without the constant term being involved for each age x . For forecasting mortality rates, the sequence $\{\hat{k}_t\}$ is assumed to follow a random walk with drift. Lee and Carter (1992) gave an approach to constructing the confidence interval for $\ln(m_{x,T})$ for age x and year T . The standard deviation of the logarithm of the projected central death rate at age x and year T ($T > t_U$), denoted by $\text{s.d.}(\ln(\hat{m}_{x,T}))$, is

$$\hat{b}_x \sqrt{\text{Var}(\hat{k}_T)},$$

where $\text{Var}(\hat{k}_T)$ is the variance of the projected k_t at time T . If $\{k_t\}$ is fitted and projected by a random walk model with $\text{Var}(k_t)$ representing the variance of the sequence of $\{k_t\}$ from time t_L to t_U in a random walk model, then $\text{Var}(\hat{k}_T) = (T - t_U)\text{Var}(k_t) = \frac{T-t_U}{t_U-(t_L+1)+1} \sum_{t=t_L+1}^{t_U} (k_t - \hat{k}_t)^2$ with \hat{k}_t as the fitted k_t in a random walk with drift model. Brouhns *et al.* (2002) proposed another approach to reaching the same result by the original idea of estimating the error term.

3.2.2 CBD model

The CBD model proposed by Cairns *et al.* (2006) is one of the non-Lee-Carter-type models. The CBD model with parameters κ_t^1 and κ_t^2 is presented as

$$\text{logit}(q_{x,t}) = \ln \left(\frac{q_{x,t}}{1 - q_{x,t}} \right) = \kappa_t^1 + \kappa_t^2(x - \bar{x}) + \epsilon_{x,t},$$

where

- the mortality data from time t_L to t_U and from age x_L to x_U are used for fitting,
- x represents each age used for mortality fitting,
- $\bar{x} = (\sum_{x_L}^{x_U} x) / (x_U - x_L + 1)$,
- the parameters κ_t^1 and κ_t^2 can be obtained by linear least squares method, and
- $\epsilon_{x,t}$ is the error term and assumed to follow a normal distribution with mean zero and to be independent of age x and time t .

To project future mortality rates, the two-dimensional time series $\{\kappa_t = (\kappa_t^1, \kappa_t^2)'\}$ is assumed to follow a random walk with drift as

$$\kappa_{t+1} = \kappa_t + \mu + C \cdot Z(t+1),$$

where μ is a 2×1 constant vector, C is a 2×2 constant upper triangular matrix, and $Z(t)$ is the two-dimensional standard normal distribution. In the model, the first factor κ_t^1 affects the mortality dynamics at all ages in the same way, whereas the second factor κ_t^2 affects the mortality dynamics at higher ages more than at lower ages.

3.3 Linear Logarithm Hazard Transform

Consider two sequences of forces of mortality (hazard rates) $\{\mu_x^E\}$ from two risk classes $E = A, B$; we assume that there is a linear relation plus an error term between $\{\mu_x^A\}$ and $\{\mu_x^B\}$ as follows:

$$\mu_x^B = \alpha_{x_L, x_U}^{A, B} \times \mu_x^A + \beta_{x_L, x_U}^{A, B} + \epsilon_x,$$

in which

- the interval $[x_L, x_U]$ is the age span used to fit this model,
- ϵ_x is the error term, assumed to follow a normal distribution with mean zero and with each ϵ_x being independent of x , and
- $\alpha_{x_L, x_U}^{A, B}$ and $\beta_{x_L, x_U}^{A, B}$ are the parameters of the simple linear regression with the mortality hazard rates for the risk class B regressed on those for the risk class A for the ages from x_L to x_U . $\hat{\alpha}_{x_L, x_U}^{A, B}$ and $\hat{\beta}_{x_L, x_U}^{A, B}$ are the estimated values of $\alpha_{x_L, x_U}^{A, B}$ and $\beta_{x_L, x_U}^{A, B}$ using the least squares method.

It is called the linear hazard transform (LHT) model because μ_x^B is a linear transform of μ_x^A .

Figure 3.1 indicates the linear relationship between three pairs of mortality hazard rate sequences: the first pair being data from 1990 against data from 1989, the second pair 1994 against 1989, and the third pair 1999 against 1989. Each sequence comprises data from ages 25 to 84. No matter whether the two sequences of mortality hazard rates being compared are one, five or ten years apart, all three scatter plots show linear trends.

Under the LHT model, the fitted survival probability $\hat{p}_x^B = (p_x^A)^{\hat{\alpha}_{x_L, x_U}^{A, B}} \times e^{-\hat{\beta}_{x_L, x_U}^{A, B}}$. In some rare cases where $\hat{\beta}_{x_L, x_U}^{A, B} < 0$ and $p_x^A \approx 1$, \hat{p}_x^B might be larger than one, a contradiction of the rule of probability. This major weakness is resolved by the linear logarithm hazard transform (LLHT) model.

Figure 3.1: Scatter plot and linear trend of the mortality hazard rates for 1990, 1994 and 1999 against the rates for 1989

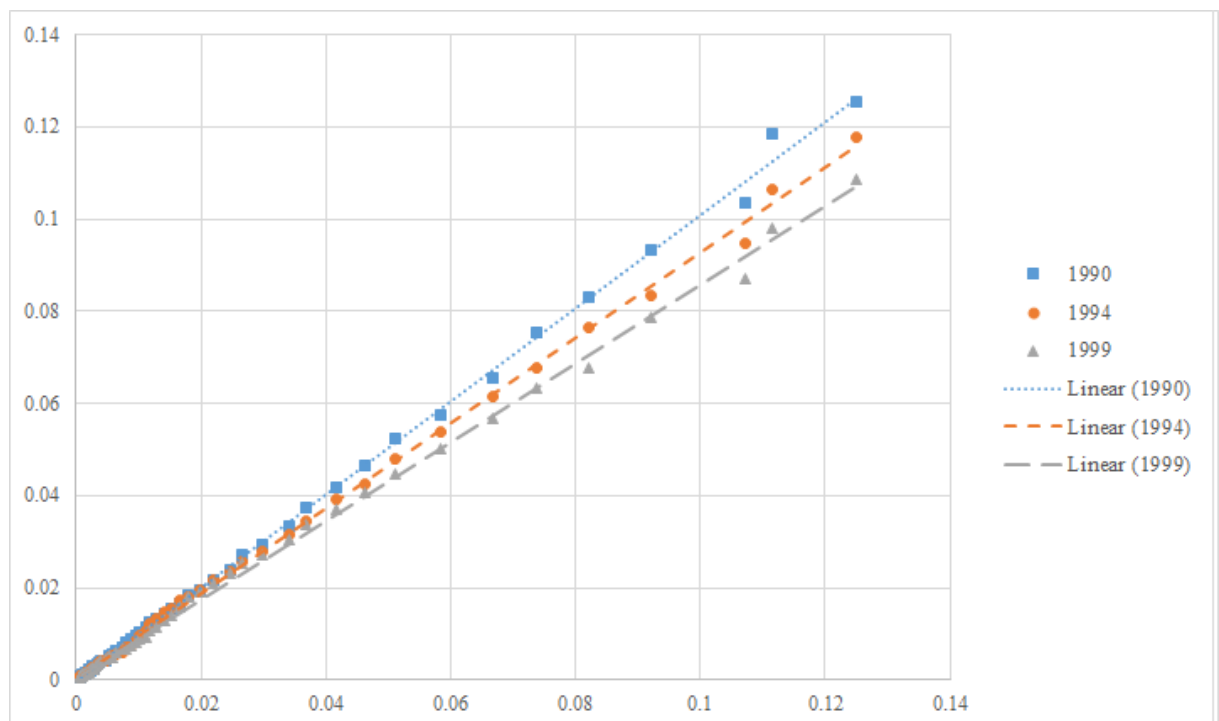
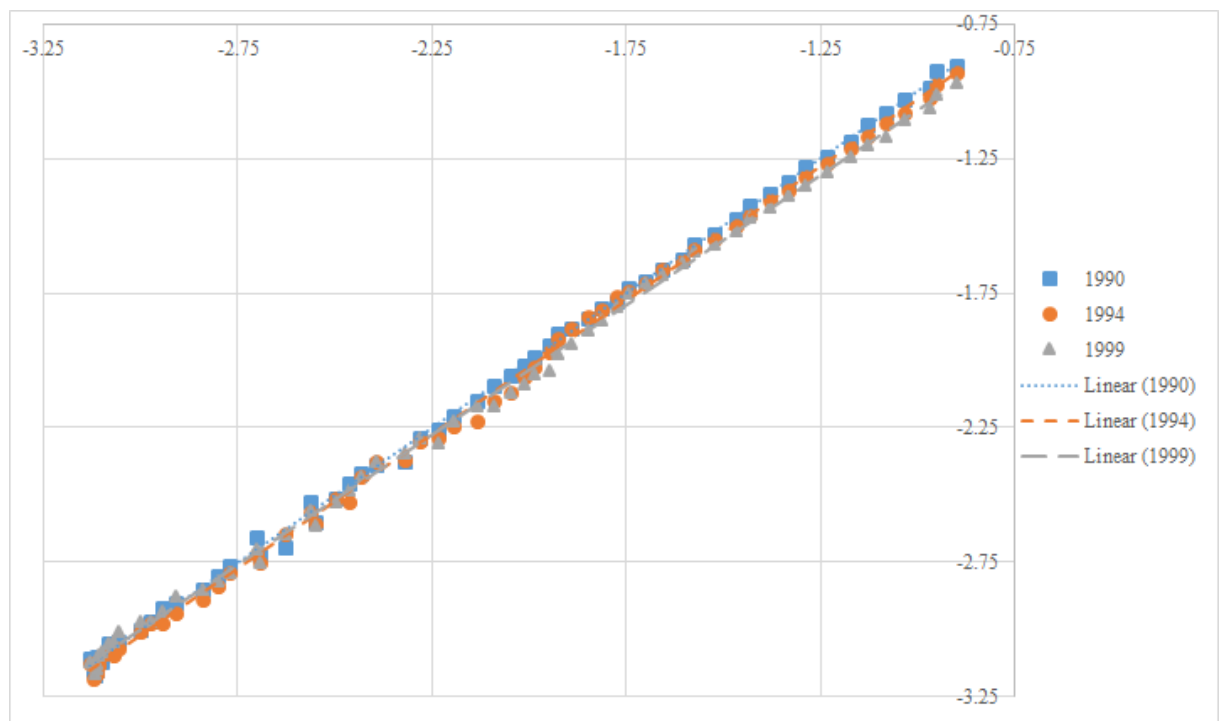


Figure 3.2: Scatter plot and linear trend of the logarithm of the mortality hazard rates for 1990, 1994 and 1999 against the rates for 1989



The LLHT model assumes that the two sequences of the natural logarithm of hazard rates have a linear relationship plus an error term as follows:

$$\ln(\mu_x^B) = \alpha_{x_L, x_U}^{A,B} \times \ln(\mu_x^A) + \beta_{x_L, x_U}^{A,B} + \epsilon_x.$$

The empirical evidence of this assumption can be seen in Figure 3.2 where the logarithm of mortality hazard rates for years 1990, 1994 and 1999 are plotted against those for year 1989. When the same assumption of constant force of mortality is made and the parameters $\alpha_{x_L, x_U}^{A,B}$ and $\beta_{x_L, x_U}^{A,B}$ are estimated by simple linear regression as usual, the fitted one-year survival probability of an individual aged x becomes

$$\hat{p}_x^B = \exp \left\{ - \left[-\ln(p_x^A) \right]^{\hat{\alpha}_{x_L, x_U}^{A,B}} \times \exp(\hat{\beta}_{x_L, x_U}^{A,B}) \right\}, \quad (3.1)$$

which ensures $\hat{p}_x^B \in (0, 1)$ though the expression is not as neat as that for the LHT model. The fitted k -year survival probability of an individual aged x would be

$${}_k\hat{p}_x^B = \exp \left\{ - \sum_{i=1}^k \left[-\ln(p_{x+i-1}^A) \right]^{\hat{\alpha}_{x_L, x_U}^{A,B}} \times \exp(\hat{\beta}_{x_L, x_U}^{A,B}) \right\}.$$

To project the mortality rates for year K from the earliest year t_L , a pair of parameters $(\hat{\alpha}_{x_L, x_U}^{t_L, K}, \hat{\beta}_{x_L, x_U}^{t_L, K})$ is needed. To simplify symbols, denote the fitted parameter using data from year t_L and year t_U , $(\hat{\alpha}_{x_L, x_U}^{t_L, t_U}, \hat{\beta}_{x_L, x_U}^{t_L, t_U})$, as $(\hat{\alpha}_{\mathbf{F}}, \hat{\beta}_{\mathbf{F}})$. Tsai and Jiang (2010) proposed the arithmetic and geometric growth methods. The arithmetic growth method assumes that the change between parameters for year t_L and year t_U is an arithmetic increment. (3.2) presents the projected parameters to predict the mortality rates for year K from year t_L , where the left subscript \mathbf{A} indicates that the parameters are projected by the arithmetic growth method.

$$(\mathbf{A}\hat{\alpha}_{x_L, x_U}^{t_L, K}, \mathbf{A}\hat{\beta}_{x_L, x_U}^{t_L, K}) = \left(1 + \frac{K - t_L}{t_U - t_L} (\hat{\alpha}_{\mathbf{F}} - 1), \frac{K - t_L}{t_U - t_L} \hat{\beta}_{\mathbf{F}} \right), \quad K = t_L, t_L + 1, \dots \quad (3.2)$$

By contrast, the geometric growth method assumes that the change between parameters from year t_L to year t_U is a geometric increment in $\hat{\alpha}_{\mathbf{F}}$. The estimated parameters linking the hazard rates in year t_L to the hazard rates in year K are then

$$(\mathbf{G}\hat{\alpha}_{x_L, x_U}^{t_L, K}, \mathbf{G}\hat{\beta}_{x_L, x_U}^{t_L, K}) = \left(\hat{\alpha}_{\mathbf{F}}^{\frac{K-t_L}{t_U-t_L}}, \frac{\hat{\alpha}_{\mathbf{F}}^{\frac{K-t_L}{t_U-t_L}} - 1}{\hat{\alpha}_{\mathbf{F}} - 1} \hat{\beta}_{\mathbf{F}} \right), \quad K = t_L, t_L + 1, \dots, \quad (3.3)$$

where the left subscript \mathbf{G} indicates that the parameters are projected by the geometric growth method. It is easy to see that both $(\mathbf{A}\hat{\alpha}_{x_L, x_U}^{t_L, K}, \mathbf{A}\hat{\beta}_{x_L, x_U}^{t_L, K})$ and $(\mathbf{G}\hat{\alpha}_{x_L, x_U}^{t_L, K}, \mathbf{G}\hat{\beta}_{x_L, x_U}^{t_L, K})$ are equal to $(1, 0)$ for $K = t_L$ and $(\hat{\alpha}_{\mathbf{F}}, \hat{\beta}_{\mathbf{F}})$ for $K = t_U$, and $\mathbf{A}\hat{\alpha}_{x_L, x_U}^{t_L, K}$ ($\mathbf{G}\hat{\alpha}_{x_L, x_U}^{t_L, K}$) grows arithmetically (geometrically) from one for $K = t_L$ to $\hat{\alpha}_{\mathbf{F}}$ for $K = t_U$. We denote the arithmetic and geometric growth methods by LLHT A and LLHT G, respectively.

The estimated force of mortality for age x and year K projected from that for the year t_L by the LLHT W method ($\mathbf{W} = \mathbf{A}, \mathbf{G}$), $\mathbf{w}\hat{\mu}_x^{t_L, K}$, is given by

$$\ln(\mathbf{w}\hat{\mu}_x^{t_L, K}) = \mathbf{w}\hat{\alpha}_{x_L, x_U}^{t_L, K} \ln(\mu_x^{t_L}) + \mathbf{w}\hat{\beta}_{x_L, x_U}^{t_L, K}. \quad (3.4)$$

Chapter 4

LLHT-based projection models

This chapter will introduce two variations of the LLHT model. In the second half of the chapter, we explore the stochastic characteristics of the LLHT model.

4.1 LLHT-based mortality projection models in a deterministic view

The arithmetic and geometric growth methods reviewed in Chapter 3 are very efficient from the aspect of the mortality data required. They use the mortality rates for only two years (the earliest and latest years) for fitting, and predict the future mortality rates with the fitted parameters. On the other hand, the Lee-Carter and the CBD models use a rectangle of the mortality data in hope to capture the historical change and to predict the upcoming changes based on the earlier trend. In this section, we use the LLHT-based mortality projection methods with a rectangle of the mortality data model like the Lee-Carter and the CBD models. Two new mortality projection approaches based on the LLHT model will be proposed to increase the accuracy of projection.

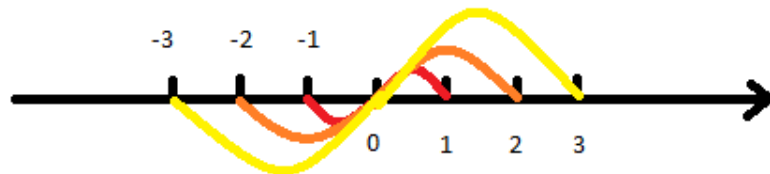
Before introducing the successful cases, we discuss some failure cases. The mortality fitting is repeated for each pair of mortality rates for two consecutive years, and a sequence of paired parameters are produced. Then the pairs of LLHT parameters are projected with a vector autoregressive model or VAR(1) and a VAR(2) model, respectively. Mathematically, a sequence of fitted parameters by the LLHT model, $\{(\hat{\alpha}_{x_L, x_U}^{t, t+1}, \hat{\beta}_{x_L, x_U}^{t, t+1}) : t = t_L, \dots, t_U - 1\}$, are obtained by fitting the logarithm of the force of mortality for year $t + 1$ with that for year t .

Then a VAR model determined by the sequence projects the sequence $\{(\hat{\alpha}_{x_L, x_U}^{t, t+1}, \hat{\beta}_{x_L, x_U}^{t, t+1}), t = t_U, t_U + 1, \dots\}$ for predicting the future mortality rates. The forecasting performance is not satisfactory according to some measuring criteria, and thus the details of this method will not be discussed further.

4.1.1 LLHT-based mortality projection variations

The first method assumes that the change in the logarithm of the force of mortality from time $-t$ to time 0 is the same as that from time 0 to time t . In other words, this method believes that the changes in the past t years can be copied to project the same duration for the future. Figure 4.1 provides a diagram of the mechanism of this method. Denote this method as LLHT C, where “C” stands for “constant”. The origin point stands for the current year t_U . More specifically, the fitted LLHT parameters $(\hat{\alpha}_{x_L, x_U}^{2t_U - K, t_U}, \hat{\beta}_{x_L, x_U}^{2t_U - K, t_U})$ is firstly obtained by fitting the logarithm of the forces of the mortality for year t_U with that for year $2t_U - K$. Then we use the pair of parameters, $(\hat{\alpha}_{x_L, x_U}^{2t_U - K, t_U}, \hat{\beta}_{x_L, x_U}^{2t_U - K, t_U})$ which is assumed to equal to $(\hat{\alpha}_{x_L, x_U}^{t_U, K}, \hat{\beta}_{x_L, x_U}^{t_U, K})$, to project the mortality rates for year K from year t_U , $K = 1, \dots, t_U - t_L$.

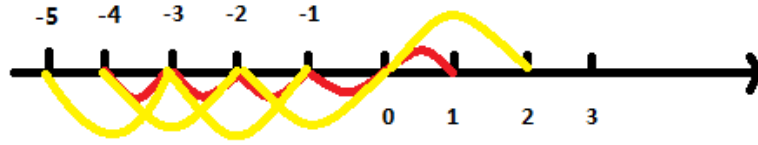
Figure 4.1: Illustration for the mortality projection method LLHT C



Another mortality projection method, denoted as LLHT T, is an advanced version of the LLHT C method because it inherits some characteristics from the LLHT C method and it also has a time effect. This method assumes that the changes in the logarithm of the forces of mortality for a sequence of the past periods of equal length can be used to predict the change in the logarithm of the force of mortality for the next period. Figure 4.2 presents

the diagram of this method for one-year mortality projection. It assumes that the historical one-year changes in $(-4,-3)$, $(-3,-2)$, $(-2,-1)$ and $(-1,0)$ can be used to project the change in the logarithm of the hazard rate for $(0,1)$, and the two-year changes in $(-5,-3)$, $(-4,-2)$, $(-3,-1)$ and $(-2,0)$ can be used to project the change in the logarithm of the hazard rate for $(0,2)$.

Figure 4.2: Illustration for the mortality projection method LLHT T



Specifically, if we want to project the mortality rates for year K from year t_U , we need the fitted LLHT parameter pair $(\hat{\alpha}_{x_L, x_U}^{t_U, K}, \hat{\beta}_{x_L, x_U}^{t_U, K})$. The sequence of the fitted LLHT parameter pairs $\{(\hat{\alpha}_{x_L, x_U}^{2t_U - K - i, t_U - i}, \hat{\beta}_{x_L, x_U}^{2t_U - K - i, t_U - i}) : i = 0, 1, \dots, n\}$ are computed by fitting the logarithm of the forces of mortality for year $t_U - i$ with that for $2t_U - k - i$, $i = 0, 1, \dots, n$, where n is set to be 9 in the numerical illustration. Then the parameter pair $(\hat{\alpha}_{x_L, x_U}^{t_U, K}, \hat{\beta}_{x_L, x_U}^{t_U, K})$ is projected by the best fitted bivariate time series model from the sequence of the LLHT parameter pairs. The ‘vars’ of R package is used in the bivariate time series fitting and projection. The VAR(p) model in the ‘vars’ of R package is given by

$$\mathbf{y}_t = A_1 \mathbf{y}_{t-1} + \dots + A_p \mathbf{y}_{t-p} + CD + \mathbf{u}_t, \quad (4.1)$$

where

- \mathbf{y}_t is a $k \times 1$ vector of endogenous variables,
- \mathbf{u}_t is a $k \times 1$ vector of spherical disturbance terms,
- CD is a $k \times 1$ constant vector, and
- the coefficients A_1, \dots, A_p are $k \times k$ matrices.

Specially in the case of the LLHT model, \mathbf{y}_t is a 2×1 vector containing the pair of LLHT parameters. The lag parameter p in the VAR model is mostly 2 and sometimes 1, and is

determined from a combination of the Akaike Information Criterion (AIC), Hannan Quinn Information Criterion (HQ), Schwarz Criterion (SC), and Akaike Final Prediction Error (FPE). Thus the customized equations for (4.1) are given by the following two equations:

$$\begin{aligned} \begin{pmatrix} \alpha_t \\ \beta_t \end{pmatrix} &= A_1 \begin{pmatrix} \alpha_{t-1} \\ \beta_{t-1} \end{pmatrix} + A_2 \begin{pmatrix} \alpha_{t-2} \\ \beta_{t-2} \end{pmatrix} + \begin{pmatrix} CD_\alpha \\ CD_\beta \end{pmatrix} + \begin{pmatrix} u_\alpha \\ u_\beta \end{pmatrix}, p = 2, \\ \begin{pmatrix} \alpha_t \\ \beta_t \end{pmatrix} &= A_1 \begin{pmatrix} \alpha_{t-1} \\ \beta_{t-1} \end{pmatrix} + \begin{pmatrix} CD_\alpha \\ CD_\beta \end{pmatrix} + \begin{pmatrix} u_\alpha \\ u_\beta \end{pmatrix}, p = 1. \end{aligned}$$

The parameter pair $(\hat{\alpha}_{x_L, x_U}^{t_U, K}, \hat{\beta}_{x_L, x_U}^{t_U, K})$ of the LLHT T method for year K is projected using the bivariate stochastic time series given above. The logarithm of the mortality hazard rate for age x and year K projected from that for year t_U using the LLHT T method, $\ln(\hat{\mu}_x^{t_U, K})$, has a form:

$$\ln(\mathbf{T}\hat{\mu}_x^{t_U, K}) = \mathbf{T}\hat{\alpha}_{x_L, x_U}^{t_U, K}(\ln(\mu_x^{t_U})) + \mathbf{T}\hat{\beta}_{x_L, x_U}^{t_U, K}, \quad (4.2)$$

where the left subscript \mathbf{T} indicates the LLHT T method.

Similarly, both the LHT C and LHT T mortality projection methods are based on the parameter pairs from the LHT mortality model. However, they are likely to produce negative mortality rates in some cases. That is why we do not cover any LHT-based deterministic mortality projection model in this project.

4.2 Confidence intervals of LLHT-based mortality projection models

The involvement of stochastic mechanism in the LLHT-based mortality projection gives the effect of uncertainty to the deterministic projection. In this section, we compute the uncertainty for the LLHT-based deterministic mortality projection models.

Before giving the details for the estimation of the variance for the stochastic LLHT model, here is the preliminary for a simple linear regression model $y_i = \beta_0 + \beta_1 x_i + \epsilon_i$, $i = 1, 2, \dots, n$. The standard deviation of the estimate of y_i has the form

$$\text{s.d.}(\hat{y}_i) = s \sqrt{\frac{1}{n} + \frac{(x_i - \bar{x})^2}{\sum_{i=1}^n (x_i - \bar{x})^2}}, \quad (4.3)$$

where $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ and $s^2 = \frac{1}{n-2} \sum_{i=1}^n (y_i - \hat{y}_i)^2$.

There is a simple linear regression between two sequences of the logarithm of mortality hazard rates according to the LLHT model. The s.d. $(\ln(\hat{\mu}_x^{t_1, t_2}))$, denoting the standard deviation of the logarithm of the fitted or projected force of mortality for year t_2 with that for year t_1 (the logarithm of the forces of mortality for years t_1 and t_2 , $\ln(\mu_x^{t_1})$ and $\ln(\hat{\mu}_x^{t_1, t_2})$, are independent and dependent variables, respectively, in the simple linear regression model), can be calculated by (4.3).

The 90% confidence interval of the one-year death probability for age x and year K is constructed upon its corresponding logarithm of the mortality hazard rate:

$$1 - \exp \left\{ \ln(\mathbf{w}\hat{p}_x^{t,K}) \times \exp \left\{ \pm t_{95\%, n-2} \times \mathbf{w}s_x^{t,K} \right\} \right\}, \quad (4.4)$$

where

- $\mathbf{w}\hat{p}_x^{t,K}$ and $\mathbf{w}\hat{\mu}_x^{t,K}$ are the estimated one-year survivor probability and mortality hazard rate, respectively, for age x and year K projected from year t ,
- W can be A, G, C or T to represent the specific mortality projection method,
- $\mathbf{w}s_x^{t,K}$ is the standard deviation of $\ln(\mathbf{w}\hat{\mu}_x^{t,K})$
- $t_{95\%, n-2}$ is the 95th percentile of Student's t-distribution with n observations. In this projection, $n = x_U - x_L + 1$.

We denote $q_{x,K}$ as the one-year death probability for age x and year K . Equation (4.4) is derived as follows.

$$\begin{aligned} \ln(-\ln(1 - q_{x,K})) &\in \ln(\mathbf{w}\hat{\mu}_x^{t,K}) \pm t_{95\%, n-2} \times \mathbf{w}s_x^{t,K} \\ \Rightarrow -\ln(1 - q_{x,K}) &\in \mathbf{w}\hat{\mu}_x^{t,K} \times \exp \left\{ \pm t_{95\%, n-2} \times \mathbf{w}s_x^{t,K} \right\} \\ \Rightarrow \ln(1 - q_{x,K}) &\in -\mathbf{w}\hat{\mu}_x^{t,K} \times \exp \left\{ \mp t_{95\%, n-2} \times \mathbf{w}s_x^{t,K} \right\} \\ &= \ln(\mathbf{A}\hat{p}_x^{t,K}) \times \exp \left\{ \mp t_{95\%, n-2} \times \mathbf{w}s_x^{t,K} \right\} \\ \Rightarrow 1 - q_{x,K} &\in \exp \left\{ \ln(\mathbf{w}\hat{p}_x^{t,K}) \times \exp \left\{ \mp t_{95\%, n-2} \times \mathbf{w}s_x^{t,K} \right\} \right\} \\ \Rightarrow q_{x,K} &\in 1 - \exp \left\{ \ln(\mathbf{w}\hat{p}_x^{t,K}) \times \exp \left\{ \pm t_{95\%, n-2} \times \mathbf{w}s_x^{t,K} \right\} \right\} \end{aligned}$$

The main topic of this section is to estimate $\mathbf{w}s_x^{t,K}$ for different LLHT-based methods.

4.2.1 Confidence interval under LLHT A and LLHT G

LLHT A

From (3.2), the logarithm of the projected force of mortality for age x and year K is

$$\begin{aligned} \ln(\mathbf{A}\hat{\mu}_x^{t_L,K}) &= \mathbf{A}\hat{\alpha}_{x_L,x_U}^{t_L,K} \times \ln(\mu_x^{t_L}) + \mathbf{A}\hat{\beta}_{x_L,x_U}^{t_L,K} \\ &= \left[1 + \frac{K-t_L}{t_U-t_L} (\hat{\alpha}_{x_L,x_U}^{t_L,t_U} - 1) \right] \times \ln(\mu_x^{t_L}) + \frac{K-t_L}{t_U-t_L} \hat{\beta}_{x_L,x_U}^{t_L,t_U}, \end{aligned}$$

where

- $\mathbf{A}\hat{\mu}_x^{t_L,K}$ is the projected force of mortality for year K from that for year t_L by LLHT A method, and
- $\hat{\alpha}_{x_L,x_U}^{t_L,K}$ and $\hat{\beta}_{x_L,x_U}^{t_L,K}$ are the estimated LLHT parameters used to project the mortality rates for year K from that for t_L . Also the subscript (x_L, x_U) indicates that the age in this projection ranges from x_L to x_U .

Assume the error term of the logarithm of the force of mortality also follows an arithmetic growth from year t_L to year t_U ; the error term for year K from year t_L , $\epsilon_x^{t_L,K}$, has a form of $\frac{K-t_L}{t_U-t_L} \epsilon_x^{t_L,t_U}$, where $\epsilon_x^{t_L,t_U}$ is the error term in the LLHT model with the mortality rates for years t_L and t_U as the fitting data. Then the standard deviation of the logarithm of the force of mortality for year K estimated from year t_L using LLHT A, $\mathbf{A}s_x^{t_L,K}$, can be expressed as $\mathbf{A}s_x^{t_L,K} \triangleq \sqrt{\text{Var}(\ln(\hat{\mu}_x^{t_L,K}))} = \frac{K-t_L}{t_U-t_L} \times \text{s.d.}(\ln(\hat{\mu}_x^{t_L,t_U}))$, $K = t_L, t_L + 1, \dots$,

The 90% confidence interval of the logarithm of the mortality hazard rate for year K and age x under the arithmetic growth method, $\ln(\mathbf{A}\hat{\mu}_x^{t_L,K})$, can be constructed as

$$\ln(\mathbf{A}\hat{\mu}_x^{t_L,K}) \pm t_{95\%,n-2} \times \mathbf{A}s_x^{t_L,K}. \quad (4.5)$$

The estimated one-year death probability for age x and year K estimated from base year t_L , $\mathbf{A}\hat{q}_x^{t_L,K}$, has the 90% confidence interval

$$1 - \exp \left\{ \ln(\mathbf{A}\hat{p}_x^{t_L,K}) \times \exp \left\{ \pm t_{90\%,n-2} \times \mathbf{A}s_x^{t_L,K} \right\} \right\}.$$

LLHT G

Constructing the 90% confidence interval of the logarithm of the mortality hazard rate $\ln(\mathbf{G}\mu_x^{t_L,K})$ for year K and age x under the geometric growth method is not as straightforward as that under the arithmetic growth method because of the complicated expressions

for $\mathbf{G}\hat{\alpha}_{x_L, x_U}^{t_L, K}$ and $\mathbf{G}\hat{\beta}_{x_L, x_U}^{t_L, K}$. Instead, the Delta method is adopted, which is an approach to estimating the variance. This project quotes the theorem in Klugman *et al.* (2012) as follows:

Theorem Let $\mathbf{X}_n = (X_{1n}, \dots, X_{kn})^T$ be a multivariate random variable of dimension k based on a sample of size n . Assume that \mathbf{X} is asymptotically normal with mean θ and covariance matrix Σ/n , where neither θ nor Σ depend on n . Let h be a function of k variables that is totally differentiable. Let $H_n = h(X_{1n}, \dots, X_{kn})$. Then \hat{H}_n is asymptotically normal with mean $h(\theta)$ and variance $(\partial \mathbf{h})^T \Sigma (\partial \mathbf{h})/n$, where $\partial \mathbf{h}$ is the vector of first derivatives, that is, $\partial \mathbf{h} = (\partial h / \partial \theta_1, \dots, \partial h / \partial \theta_k)^T$ and it is to be evaluated at θ , the true parameters of the original random variable.

Taking the $(\hat{\alpha}_{\mathbf{F}}, \hat{\beta}_{\mathbf{F}})$ in (3.3) as two random variables, function h can be written as

$$h(\hat{\alpha}_{\mathbf{F}}, \hat{\beta}_{\mathbf{F}}) \triangleq \ln(\hat{\mu}_x^{t_L, K}) = \hat{\alpha}_{\mathbf{F}}^{\frac{K-t_L}{t_U-t_L}} [\ln(\mu_x^{t_L})] + \frac{\hat{\alpha}_{\mathbf{F}}^{\frac{K-t_L}{t_U-t_L}} - 1}{\hat{\alpha}_{\mathbf{F}} - 1} \hat{\beta}_{\mathbf{F}} \quad (4.6)$$

based on (3.4). Rewrite formula (4.6) in matrix notation as

$$H(\hat{\alpha}_{\mathbf{F}}, \hat{\beta}_{\mathbf{F}}) = Z(\hat{\alpha}_{\mathbf{F}}^{\frac{K-t_L}{t_U-t_L}}, \frac{\hat{\alpha}_{\mathbf{F}}^{\frac{K-t_L}{t_U-t_L}} - 1}{\hat{\alpha}_{\mathbf{F}} - 1} \hat{\beta}_{\mathbf{F}})^T,$$

where Z is the matrix given by

$$Z = \begin{bmatrix} \ln(\mu_{x_L}^{t_L}) & 1 \\ \vdots & \vdots \\ \ln(\mu_{x_U}^{t_L}) & 1 \end{bmatrix}.$$

The derivative of function h , $\partial h(\hat{\alpha}_{\mathbf{F}}, \hat{\beta}_{\mathbf{F}}) = (\frac{\partial h}{\partial \hat{\alpha}_{\mathbf{F}}}, \frac{\partial h}{\partial \hat{\beta}_{\mathbf{F}}})^T$, is given below

$$\frac{\partial h}{\partial \hat{\alpha}_{\mathbf{F}}} = \frac{K-t_L}{t_U-t_L} \hat{\alpha}_{\mathbf{F}}^{\frac{K-t_L}{t_U-t_L}-1} [\ln(\mu_x^{t_L})] + \frac{(\frac{K-t_L}{t_U-t_L} - 1) \hat{\alpha}_{\mathbf{F}}^{\frac{K-t_L}{t_U-t_L}} - \frac{K-t_L}{t_U-t_L} \hat{\alpha}_{\mathbf{F}}^{\frac{K-t_U}{t_U-t_L}-1} + 1}{(1 - \hat{\alpha}_{\mathbf{F}})^2} \hat{\beta}_{\mathbf{F}} \quad (4.7)$$

and

$$\frac{\partial h}{\partial \hat{\beta}_{\mathbf{F}}} = \frac{1 - \hat{\alpha}_{\mathbf{F}}^{\frac{K-t_L}{t_U-t_L}}}{1 - \hat{\alpha}_{\mathbf{F}}}. \quad (4.8)$$

Next, the covariance matrix of these two random variables $(\hat{\alpha}_{\mathbf{F}}, \hat{\beta}_{\mathbf{F}})$ is

$$\Sigma/n = \text{Cov}(\hat{\alpha}_{\mathbf{F}}, \hat{\beta}_{\mathbf{F}}) = s^2(Z'Z)^{-1}, \quad (4.9)$$

where $s^2 = \frac{1}{n-2}(y-\hat{y})^T(y-\hat{y})$, $y = (\ln(\mu_{x_L}^{t_L}), \dots, \ln(\mu_{x_U}^{t_L}))$ is an $n \times 1$ vector of the logarithm of mortality hazard rates observed at time t_L , $\hat{y} = (\ln(\hat{\mu}_{x_L}^{t_L, K}), \dots, \ln(\hat{\mu}_{x_U}^{t_L, K}))$ is the vector of the logarithm of the projected force of mortality for year K from year t_L , and $n = x_U - x_L + 1$. With (4.7), (4.8) and (4.9), the standard deviation of the logarithm of the projected force of mortality for year K from year t_L , $\mathbf{G}s_x^{t_L, K} \triangleq \sqrt{\text{Var}[\ln(\hat{\mu}_x^{t_L, K})]}$, $K = t_L, t_L + 1, \dots$, can be achieved by the theorem for the delta method. Then the 90% confidence interval of $\mathbf{G}q_x^K$ for year K and age x under the geometric growth method is

$$1 - \exp \left\{ \ln(\mathbf{G}\hat{p}_x^{t_L, K}) \times \exp \left\{ \pm t_{95\%, n-2} \times \mathbf{G}s_x^{t_L, K} \right\} \right\}.$$

4.2.2 Confidence interval under LLHT C

Recall the assumption for the LLHT C method where the change in the logarithm of the force of mortality from time 0 to time t is the same as that from time $-t$ to time 0. In this case, the estimated variance of the logarithm of the force of mortality for time t under the LLHT C method has the same value as the variance estimated by fitting the logarithm of the force of mortality for time 0 with that for time $-t$. Now the current year is t_U (time 0) and we forecast the mortality rates for year $K > t_U$. Based on the assumption above, we have $\mathbf{C}s_x^{t_U, K} \triangleq \text{s.d.}[\ln(\mu_x^{t_U, K})] = \text{s.d.}[\ln(\mu_x^{2t_U - K, t_U})]$ for $K = t_U + 1, \dots, 2 \cdot t_U - t_L$ where $\text{s.d.}[\ln(\mu_x^{2t_U - K, t_U})]$ can be calculated by (4.3). Then the 90% confidence interval of $\mathbf{C}q_x^K$ for year K and age x under the LLHT C method is

$$1 - \exp \left\{ \ln(\mathbf{C}\hat{p}_x^{t_U, K}) \times \exp \left\{ \pm t_{95\%, n-2} \times \mathbf{C}s_x^{t_U, K} \right\} \right\}.$$

4.2.3 Confidence interval under LLHT T

The calculation of the variance of the logarithm of the mortality hazard rate for age x and year K estimated from that for year t_U under the LLHT T method, $\text{Var}[\ln(\mathbf{T}\hat{\mu}_x^{t_U, K})]$, is more complicated than that under the LLHT C method. First, the variance of $\ln(\mathbf{T}\hat{\mu}_x^{t_U, K})$ can be derived from (4.2) as

$$\text{Var}[\ln(\mathbf{T}\hat{\mu}_x^{t_U, K})] = [\ln(\mu_x^{t_U})]^2 \text{Var}[\hat{\alpha}_{x_L, x_U}^{t_U, K}] + 2 \ln(\mu_x^{t_U}) \text{Cov}[\hat{\alpha}_{x_L, x_U}^{t_U, K}, \hat{\beta}_{x_L, x_U}^{t_U, K}] + \text{Var}[\hat{\beta}_{x_L, x_U}^{t_U, K}].$$

Estimating the variance-covariance matrix of $\hat{\alpha}_{x_L, x_U}^{t_U, K}$ and $\hat{\beta}_{x_L, x_U}^{t_U, K}$ is the key to calculate $\text{Var}[\ln(\mathbf{T}\hat{\mu}_x^{t_U, K})]$. We utilize the ‘Psi’ function in R package ‘vars’ to calculate this variance-covariance matrix of $\hat{\alpha}_{x_L, x_U}^{t_U, K}$ and $\hat{\beta}_{x_L, x_U}^{t_U, K}$.

Finally, denoting $\mathbf{T}s_x^{t_U, K}$ as the standard deviation of $\ln(\hat{\mu}_x^{t_U, K})$ by the LLHT T method, the 90% confidence interval of the one-year death probability for year K and age x can be represented as

$$1 - \exp \left\{ \ln(\mathbf{T}\hat{p}_x^{t_U, K}) \times \exp \left\{ \pm t_{95\%, n-2} \times \mathbf{T}s_x^{t_U, K} \right\} \right\}.$$

Chapter 5

Valuation of mortality-linked securities

In this chapter, we study some simple mortality-linked securities including q-forward and modified q-forward. To make this study simpler, only the mortality decrement and interest discount are considered in all applications of this project. There are no expenses and other decrements like lapse involved in the discussion.

5.1 q-forward

J.P. Morgan issued a zero-coupon mortality swap product called *q*-forward in 2007. Mortality swaps, sometimes called survivor swaps, were described by Dowd *et al.* (2006) as an agreement between two firms to swap a fixed amount with a random amount at some future time t , where one or both amounts are linked to some mortality experience. Coughlan *et al* (2007) stated that J.P. Morgan's q-forward is a standardized mortality swap that can be purchased by either a life insurance company or an annuity provider. Two counterparties (J.P. Morgan and another company) in this contract will swap two cash flows based on the pre-determined mortality rates and the realized mortality rates at a defined date (see Figure 5.1). Some characteristics of q-forward are listed as follows:

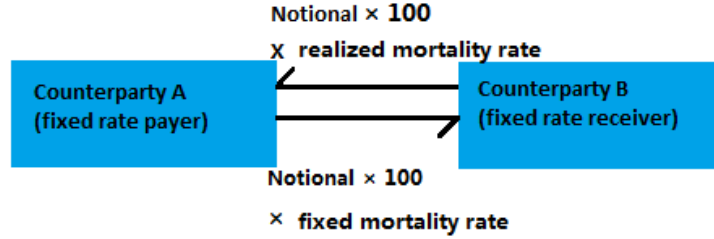
- The q-forward is a standardized contract between J.P. Morgan and the company that wants to hedge mortality or longevity risk.

- One of the key elements in this product is that the fixed mortality rate (or forward mortality rate) is set to be below the “best estimate” of the underlying mortality rate.
- Investors require a premium to cover the spread between the forward mortality rate and the “best estimate” mortality rate.
- LifeMetrics is adopted as the reference mortality rate. Currently, LifeMetrics is only available for the USA, England and Wales, Netherlands and Germany.

According to Coughlan *et al* (2007), counterparty A and counterparty B in Figure 5.1 are different regarding the participation of a life insurer or an annuity provider. If a life insurance company is the participant, then J.P. Morgan is the forward (or fixed) mortality rate receiver while the life insurance company is the floating mortality rate receiver. Life insurance companies are hedging for the mortality risk (i.e. the risk of realized mortality rates being higher than expected). If the realized mortality rates are higher than expected, insurance companies will lose money because they pay out the death benefits earlier than expected. The life insurance company in the q-forward contract will receive the net settlement based on the realized mortality rate and forward mortality rate. If the realized mortality rate is higher than the forward mortality rate, the insurance company will receive a positive settlement. On the other hand, an annuity provider bears more burden with the longevity risk (i.e. the risk of realized mortality rates being lower than expected), and thus is the fixed mortality rate receiver. If the realized mortality rate is lower than expected, the annuity provider will lose money because they pay out the annuity longer than expected. The annuity provider in the q-forward contract will receive the net settlement based on the realized mortality rates and forward mortality rates as well. If the realized mortality rate is lower than the forward mortality rate, the annuity provider will receive a positive settlement.

Denote $q_{x+T,T}^z$ the mortality rate set for age x at issue and age $x + T$ at maturity time T with right superscript $z = f, r$ and e as the forward (or fixed), realized and “best estimate” mortality rates, respectively. The floating mortality rate or the realized mortality rate is defined as the mortality rate proportional to the LifeMetrics Index. This is a simple transform for the projected mortality rate and is the method used for pricing the q-forward in this project. If both counterparties agree on the notional amount F in the contract, the life insurer, which is the floating mortality rate taker in the q-forward scheme, is expecting

Figure 5.1: A q-forward exchange illustration



a settlement of

$$F \times (q_{x+T,T}^r - q_{x+T,T}^f) \tag{5.1}$$

at the maturity time T , while the settlement for the annuity provider is expected to be

$$F \times (q_{x+T,T}^f - q_{x+T,T}^r) \tag{5.2}$$

since the annuity provider is the fixed mortality rate receiver. Coughlan *et al.* (2007) argued that the fixed mortality rate can be calculated as

$$q_{x+T,T}^f = (1 - T \times \lambda \times \sigma) \times q_{x+T,T}^e, \tag{5.3}$$

given the required Sharpe ratio λ with the notation $q_{x+T,T}^e$ as the “best estimate” mortality rate and σ as the standard deviation of the mortality rate derived from the historical mortality data. Since its revision by Sharpe (1994), the Sharpe ratio is defined as:

$$\lambda = \frac{E[R_a - R_b]}{\sqrt{\text{Var}[R_a - R_b]}}$$

where R_a is the return of the underlying asset and R_b is the return on a benchmark asset, (such as the risk-free rate of return or an index, e.g., the S&P 500). We assume the required Sharpe ratio $\lambda = 0.25$, the same value used by Li and Hardy (2011).

The fixed mortality rate for age x at issue and age $x + T$ at the maturity time T of the q-forward, $q_{x+T,T}^f$, can be accessed by (5.3). The fixed mortality rate receiver in this contract will be expecting to receive a risk premium of $F \times (q_{x+T,T}^e - q_{x+T,T}^f) \times v^T$ which

can be rewritten as

$$\begin{aligned} F \times (q_{x+T,T}^e - q_{x+T,T}^f) \times v^T &= F \times [q_{x+T,T}^e - (1 - T \times \lambda \times \sigma) \times q_{x+T,T}^e] \times v^T \\ &= F \times (T \times \lambda \times \sigma) \times q_{x+T,T}^e \times v^T. \end{aligned} \quad (5.4)$$

5.2 Modified q-forward

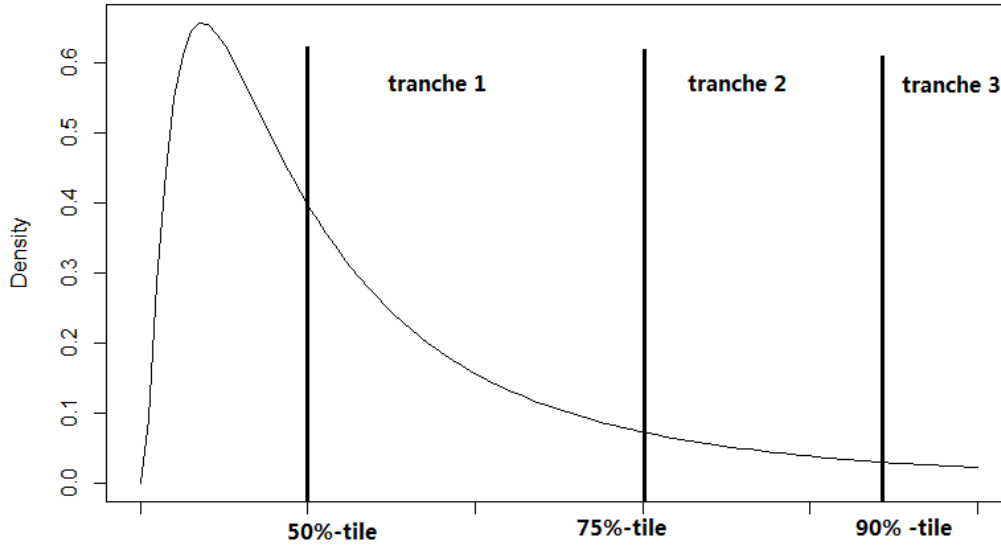
For the modified q-forward, credit tranche technique is applied to creating a specific yield-risk relationship, where the yield is higher if the underlying rate is more extreme. This product makes insurance companies or annuity providers take the majority of the base mortality fluctuation as legitimate business risk and transfer the more severe risks to the special purpose vehicle (SPV) which issues the modified q -forward. Under this setup, the SPV always takes the floating mortality rate in each tranche in the contrast with either a life insurer or an annuity provider, in contrast to the relationship between a life insurance company or an annuity provider and J.P. Morgan under the q -forward contract.

The tranche design is illustrated in Figure 5.2. Assume that the mortality rate has a density curve as that in Figure 5.2. Tranches 1, 2 and 3 contain the layers between the 50th and 75th percentiles, the 75th and 90th percentiles, and the 90th and 100th percentiles of the mortality rate distribution, respectively. Overall, this tranche design serves to hedge the floating mortality rate higher than the median, which is ideal for insurance companies looking to hedge the risk of earlier death benefit payments. According to the probability assigned to each tranche, Tranche 1 contains the biggest portion. However, Tranche 1 is the closest tranche to the base point, the 50th percentile. Thus, the payment on the excess loss for Tranche 1 is the smallest among all three.

There are two approaches to determining the amount of reimbursement rates.

1. The reimbursement depends on the tranche that the realized mortality rate falls into. For example, if the realized mortality rate falls into the second tranche, the reimbursement amount is simply the product of a predetermined rate for the second tranche and the realized mortality rate less the specified base mortality rate applicable to that tranche (i.e. the 75th percentile of the distribution of the underlying mortality rate). However, this approach will encourage false reporting of the realized mortality because the reimbursement will have a jump at the detachment point of the tranche.

Figure 5.2: Credit tranche technique illustration



2. To avoid false reporting, the excess loss is applied to eliminate the jump at the detachment point of the tranche.

The following four steps summarize the pricing of the modified q-forward with a life insurance company using the excess loss method.

1. The detachment point and the risk for each tranche can be “tailor-made”. Tranche 1 contains the 50th to 75th percentile of the mortality rate distribution, Tranche 2 contains the 75th to 90th percentile of the mortality rate distribution, and Tranche 3 contains the 90th to 100th percentile of the mortality rate distribution. The SPV takes $r^{(1)} = 1\%$, $r^{(2)} = 10\%$ and $r^{(3)} = 100\%$ of the excess mortality rate for Tranches 1, 2 and 3, respectively.
2. The excess loss for Tranche j at time t for an individual aged x at issue, $L_{x+t}^{(j)}$, can be defined as

$$\begin{aligned} L_{x+t}^{(j)} &= \text{Min} \left\{ (q_{x+t}^{\lambda^{(j)}} - q_{x+t}^{\lambda^{(j-1)}}), \text{Max} \left\{ 0, q_{x+t,t} - q_{x+t}^{\lambda^{(j-1)}} \right\} \right\} \\ &= [q_{x+t,t} - q_{x+t}^{\lambda^{(j-1)}}]_+ - [q_{x+t,t} - q_{x+t}^{\lambda^{(j)}}]_+, \end{aligned}$$

where $q_{x+t}^{\lambda(j-1)}$ and $q_{x+t}^{\lambda(j)}$ are the lower and upper detachment points of the j th tranche, respectively, $q_{x+t,t}$ is the realized mortality rate at time t for an individual aged x at time 0, and

$$(x - x^{(j)})_+ = \begin{cases} x - x^{(j)}, & \text{if } x > x^{(j)}, \\ 0, & \text{otherwise.} \end{cases}$$

In our case, let $q_{x+t}^{\lambda(0)} = 50^{\text{th}}$ percentile, $q_{x+t}^{\lambda(1)} = 75^{\text{th}}$ percentile, $q_{x+t}^{\lambda(2)} = 90^{\text{th}}$ percentile and $q_{x+t}^{\lambda(3)} = 100^{\text{th}}$ percentile of the distribution of the mortality rate q_{x+t} for age $x + t$. According to the tranche design above, for a realized mortality rate $q_{x+t,t}^{80\%}$ (the 80^{th} percentile of the distribution of the mortality rate at year t) falling in the second tranche, the excess losses for the first and second tranches are $(q_{x+t}^{\lambda(1)} - q_{x+t}^{\lambda(0)})$ and $(q_{x+t,t}^{80\%} - q_{x+t}^{\lambda(1)})$, respectively.

3. The SPV pays a rate $r^{(j)}$ for the excess loss in Tranche j such that the price for year t is

$$R_{x+t}^{(j)} = F \times r^{(j)} \times L_{x+t}^{(j)}, \quad (5.5)$$

where $r^{(j)}$ is the percent of the excess loss paid by the SPV, and F is the face amount or the death benefit of the life insurance. For instance, in the example above the SPV will pay the total amount equal to $F \times [r^{(1)} \times (q_{x+t}^{\lambda(1)} - q_{x+t}^{\lambda(0)}) + r^{(2)} \times (q_{x+t,t}^{80\%} - q_{x+t}^{\lambda(1)})]$ at year t .

4. The total premium at year zero for the contract maturing at time t is

$$v^t \sum_j E \left[R_{x+t}^{(j)} \right] = F \times v^t \sum_j r^{(j)} \times E(L_{x+t}^{(j)}). \quad (5.6)$$

For an annuity provider, the tranches are on the left-hand side of the distribution of the underlying mortality rate because the annuity provider is more concerned about lower mortality rate which leads to annuity payments longer than expected. As a result, the tranches are assigned $q_{x+t}^{\lambda(0)} = 50^{\text{th}}$ percentile, $q_{x+t}^{\lambda(1)} = 25^{\text{th}}$ percentile, $q_{x+t}^{\lambda(2)} = 10^{\text{th}}$ percentile and $q_{x+t}^{\lambda(3)} = 0^{\text{th}}$ percentile of the mortality rate distribution for age $x + t$. The excess loss for Tranche j at time t for an individual aged x at issue, $L_{x+t}^{(j)}$, can be defined as

$$\begin{aligned} L_{x+t}^{(j)} &= \text{Min}[(q_{x+t}^{\lambda(j-1)} - q_{x+t}^{\lambda(j)}), \text{Max}(0, q_{x+t}^{\lambda(j-1)} - q_{x+t,t})] \\ &= [q_{x+t}^{\lambda(j-1)} - q_{x+t,t}]_+ - [q_{x+t}^{\lambda(j)} - q_{x+t,t}]_+. \end{aligned} \quad (5.7)$$

The steps of pricing the modified q-forward for a contract with an annuity provider are the same as those for pricing with a life insurer, but the expectation of the excess loss is estimated differently. We discuss the expectation of the excess loss for an annuity provider in the next subsection.

5.2.1 Expectation of excess loss

To evaluate (5.6) for life insurers, we need to calculate the expected value of the excess loss for each tranche,

$$E(L_{x+t}^{(j)}) = E \left[(q_{x+t,t} - q_{x+t}^{\lambda(j-1)})_+ \right] - E \left[(q_{x+t,t} - q_{x+t}^{\lambda(j)})_+ \right],$$

or

$$E(L_{x+t}^{(j)}) = E \left[(e^{-U_{x+t}^{(j-1)}} - e^{-U_{x+t,t}})_+ \right] - E \left[(e^{-U_{x+t}^{(j)}} - e^{-U_{x+t,t}})_+ \right],$$

where U is the central death rate under the Lee-Carter model or the force of mortality under the LLHT model with $U^{(j)}$ and $U^{(j-1)}$ as the corresponding values of the upper and lower detachment points under Tranche j , respectively, $U_{x+t,t}$ represents the central death rate or the force of mortality for age $x+t$ and year t , and $U_{x+t}^{(j)}$ is the central death rate or the force of mortality for age $x+t$ for the upperbound under Tranche j . Given the fact that the logarithm of the central death rate under the Lee-Carter model and the logarithm of the force of mortality under the LLHT model have a normal distribution, we can say that U follows a lognormal distribution. If $U \sim \text{lognorm}(\mu, \sigma^2)$, the expectation of the excess loss,

$$E \left[(e^{-U^{(j)}} - e^{-U})_+ \right] = \int_{U^{(j)}}^{\infty} (e^{-U^{(j)}} - e^{-u}) \frac{1}{u\sqrt{2\pi\sigma^2}} e^{-\frac{(\ln u - \mu)^2}{2\sigma^2}} du.$$

The integration above is not easy to compute. The mathematical package Maple is used to get the numerical result of this integration.

The definition of the excess loss for annuity providers is given in (5.7), and its expectation can be computed as

$$\begin{aligned} E(L_{x+t}^{(j)}) &= E \left[(q_{x+t}^{\lambda(j-1)} - q_{x+t,t})_+ \right] - E \left[(q_{x+t}^{\lambda(j)} - q_{x+t,t})_+ \right] \\ &= E \left[(e^{-U_{x+t,t}} - e^{-U_{x+t}^{(j-1)}})_+ \right] - E \left[(e^{-U_{x+t,t}} - e^{-U_{x+t}^{(j)}})_+ \right]. \end{aligned} \quad (5.8)$$

Taking the second term on the right-hand side of (5.8) as an illustration,

$$E[(e^{-U} - e^{-U^{(j)}})_+] = \int_0^{U^{(j)}} (e^{-u} - e^{-U^{(j)}}) f_U(u) du$$

where f_U is the density function of the random variable U which is assumed to have a lognormal distribution. This integration is also calculated using Maple.

5.2.2 Calculations of the detachment points of tranches

The log-normal distribution has an important characteristic that connects to the normal distribution: if a random variable X follows a normal distribution, then the random variable $Y = e^X$ has a log-normal distribution with the same parameters. Thus, if $x_{0.4}$ is the 40th percentile of X , then $y_{0.4} = e^{x_{0.4}}$ is the 40th percentile of Y . Now U has a log-normal distribution with parameters (μ, σ^2) and $\ln(U)$ has a normal distribution with parameters (μ, σ^2) . Therefore,

$$\ln(U^{(j)}) = \mu + z^{(j)}\sigma$$

is the location of the $(j + 1)$ st detachment point for $\ln(U)$ where $z^{(j)}$ is the corresponding percentile of the standard normal distribution, implying that the location of the $(j + 1)$ st detachment point for q is

$$q^{(j)} = 1 - e^{-U^{(j)}}$$

since the mortality rate q is an increasing function of U .

Chapter 6

Numerical illustrations

This chapter compares the accuracy of the mortality rates projected by the Lee-Carter model, the CBD model and the LLHT-based models. Also, the pricing of two mortality-linked securities is illustrated.

The time series $\{k_t\}$ in the Lee-Carter model and both time series $\{\kappa_t^1\}$ and $\{\kappa_t^2\}$ in the CBD mortality models are assumed to follow a random walk with drift. The age span $[x_L, x_U] = [25, 84]$ and year span $[t_L, t_U] = [1989, 1999]$ from the Human Mortality Database (2013) are selected for the mortality fitting. The mortality rates for the next ten years $[2000, 2009]$ are forecasted based on the fitted parameters, and the forecasted mortality rates are compared for accuracy with the actual ones for the same year span $[2000, 2009]$.

6.1 Illustrations for mortality projections in the deterministic view

6.1.1 Illustration for the LLHT C method

Table 6.1 presents the parameter pair (α, β) for Japanese males and females needed for mortality projection by the LLHT C method. The parameters α and β are estimated by two sequences of mortality rates of Japanese males for year 1999 and year 1999 $- K$. For example, if $K = 1$, then the parameters $\alpha = 0.997552$ and $\beta = -0.003075$ by the LLHT model contain the information about improvement or deterioration of one-year mortality rates. The year 2000 for projection has one year distance from year 1999. As a result,

the parameters estimated based on year 1998 and year 1999 are applied to estimate the mortality rates for year 2000. Therefore, the projected one-year survival probability for Japanese males for year 2000 is $\hat{p}_x^{2000,M} = \exp\{-[-\ln(-\ln(p_x^{1999,M}))]^{0.997552} \times e^{0.003075}\}$ by (3.1) where $-\ln(p_x^{1999,M})$ is the observed one-year survival probability for year 1999.

Table 6.1: Projected parameters for Japan with the LLHT C method

base year 1999	Japan, Male		Japan, Female	
forecast year	α	β	α	β
2000	0.997552	-0.003075	1.003386	0.014077
2001	0.975985	-0.088976	0.988753	-0.071661
2002	0.975658	-0.106761	0.976484	-0.157387
2003	0.968636	-0.183036	0.971488	-0.243326
2004	0.968597	-0.167939	0.964434	-0.260981
2005	0.959012	-0.240350	0.949303	-0.379733
2006	0.962664	-0.233049	0.956841	-0.358211
2007	0.961855	-0.238149	0.952447	-0.390168
2008	0.961914	-0.252095	0.949263	-0.431907
2009	0.966401	-0.234337	0.953490	-0.427168

All the projections by the LLHT C method are from year 1999. The parameters in Table 6.1 reveal a compelling evidence of mortality improvement. When the projection year increases, the values of α and β show a trend of decrease for both the male and female populations of Japan. Therefore, the logarithm of the forecasted forces of mortality decreases year by year, or equivalently, the mortality rates are expected to improve from year 1999.

6.1.2 Illustration for the LLHT T method

Based on the LLHT T method, to forecast mortality rates for year $1999 + K$ from year 1999, the sequence of the LLHT parameter pairs $\{(\alpha^{1999-K-i,1999-i}, \beta^{1999-K-i,1999-i}) : i = 0, 1, \dots, 9\}$ are first estimated. The sequence of the LLHT parameter pairs of length 10 is then fitted by a bivariate time series model to project the LLHT parameter pairs $(\alpha^{1999,1999+K}, \beta^{1999,1999+K})$ for year $1999 + K$ which is K years away from year 1999. Table 6.2 gives all the forecasted parameter pairs needed to project the mortality rates for

2000, ..., 2009. For example, for $K = 1$, the projected one-year survival probability for Japanese males for year 2000 is $\hat{p}_x^{2000,M} = \exp\{-\ln[-\ln(p_x^{1999,M})]^{1.005670} \times e^{0.001775}\}$.

Table 6.2: Projected parameters for Japan with the LLHT C method

base year 1999	Japan, male		Japan, female	
forecast year	α	β	α	β
2000	1.005670	0.001775	0.996214	-0.051023
2001	1.021356	0.013127	0.984679	-0.175274
2002	0.997485	-0.044667	0.973228	-0.248232
2003	0.992938	-0.103491	0.981298	-0.218047
2004	1.004238	-0.093128	0.979622	-0.237523
2005	0.998783	-0.132927	0.978698	-0.299949
2006	1.003713	-0.213816	0.993692	-0.295633
2007	1.004612	-0.154192	0.972744	-0.360127
2008	0.998856	-0.229689	0.969577	-0.439333
2009	0.993044	-0.302176	0.973502	-0.473026

The values of α in Table 6.2 do not show as clear a trend of decrease as those under in the LLHT C model for the male and female populations of Japan, whereas a trend of decrease can be seen from the values of β as the projection year increases.

6.1.3 Accuracy of the projected mortality rates

Figure 6.1 displays plots of the true and deterministically projected mortality rates with different mortality models. Japanese males and females aged 30, 50, 70 are selected to represent the young, middle and old populations, respectively. Although the accuracy of the projections varies by each of these three age groups, these plots still can reveal some properties. First, the projected mortality rates from the Lee-Carter, CBD, LLHT A and LLHT G methods look approximately linear, whereas the true mortality rates and those from the LLHT C and LLHT T methods do not look linear. Second, from the plot for true mortality rates (black dot line), the mortality improvement for age 70 is more severe than that for age 50 for both males and females of Japan while the group representing the younger male population of Japan (age 30) shows the least evidence of mortality improvement. From the six plots in Figure 6.1, the LLHT T method seems to present the overall best results

among the six mortality methods because its predicted mortality rates are closest to the true mortality rates.

To examine and compare the accuracy of the mortality projection from the different mortality projection methods, three statistics are adopted to measure the errors. These three statistics, the root mean square deviation (RMSE), the mean absolute error (MAE) and the mean absolute percentage error (MAPE), are to measure the deviation of the projected mortality rates from the real mortality rates. Let $\hat{q}_{x,K}$ and $q_{x,K}$ denote the projected and real mortality rates for age x and year K , respectively. Then $RMSE_K$, MAE_K and $MAPE_K$ over the age span $[x_L, x_U]$ for year K are given by

$$RMSE_K = \sqrt{\frac{1}{x_U - x_L + 1} \sum_{x=x_L}^{x_U} [\hat{q}_{x,K} - q_{x,K}]^2},$$

$$MAE_K = \frac{1}{x_U - x_L + 1} \sum_{x=x_L}^{x_U} |\hat{q}_{x,K} - q_{x,K}|,$$

and

$$MAPE_K = \frac{1}{x_U - x_L + 1} \sum_{x=x_L}^{x_U} \left| \frac{\hat{q}_{x,K} - q_{x,K}}{q_{x,K}} \right| \times 100\%. \quad (6.1)$$

The $RMSE_x$, MAE_x and $MAPE_x$ over the year span $[t_U + 1, t_U + m]$ for age x are defined by

$$RMSE_x = \sqrt{\frac{1}{m} \sum_{K=t_U+1}^{t_U+m} [\hat{q}_{x,K} - q_{x,K}]^2},$$

$$MAE_x = \frac{1}{m} \sum_{K=t_U+1}^{t_U+m} |\hat{q}_{x,K} - q_{x,K}|,$$

and

$$MAPE_x = \frac{1}{m} \sum_{K=t_U+1}^{t_U+m} \left| \frac{\hat{q}_{x,K} - q_{x,K}}{q_{x,K}} \right| \times 100\%. \quad (6.2)$$

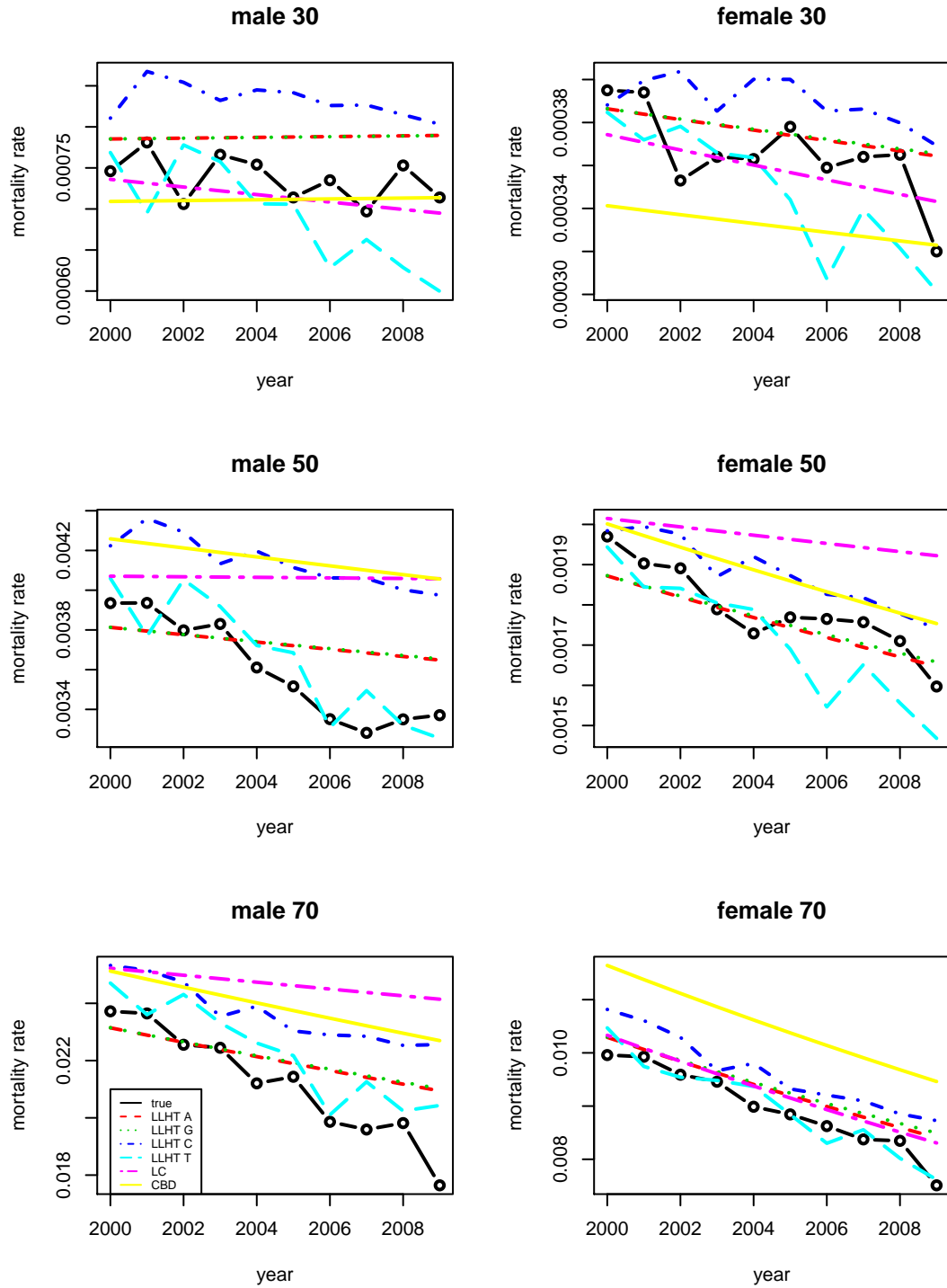
The overall MAE and $MAPE$ over the year span $[t_U + 1, t_U + m]$ and the age span $[x_L, x_U]$ are defined as

$$MAE = \frac{1}{m} \sum_{K=t_U+1}^{t_U+m} MAE_K = \frac{1}{x_U - x_L + 1} \sum_{x=x_L}^{x_U} MAE_x$$

and

$$MAPE = \frac{1}{m} \sum_{K=t_U+1}^{t_U+m} MAPE_K = \frac{1}{x_U - x_L + 1} \sum_{x=x_L}^{x_U} MAPE_x,$$

Figure 6.1: Deterministic mortality projections for ages 30, 50 and 70 of Japan



respectively. Depending on whether the sum of the squared errors is calculated with respect to age or or year first, the RMSE can be defined as

$$RMSE = \frac{1}{m} \sum_{K=t_U+1}^{t_U+m} RMSE_K$$

or

$$RMSE = \frac{1}{x_U - x_L + 1} \sum_{x=x_L}^{x_U} RMSE_x.$$

In this project, we adopt the former. Both RMSE and MAE measure the absolute deviation, while MAPE measures the relative deviation. A small MAE may result in a large MAPE when the real mortality rates (the denominator) are very small, as they are at the young ages.

Table 6.3: Projection errors for the age span [25, 80] and the year span [2000, 2009]

country/gender	error	LLHT A	LLHT G	LLHT C	LLHT T	LC	CBD
Japan male	RMSE	19.14	19.68	16.99	14.32	33.09	25.19
	MAE	11.13	11.45	10.74	7.54	19.21	14.51
	MAPE	9.23	9.35	10.42	5.80	9.90	13.93
Japan female	RMSE	9.12	9.79	7.33	5.89	6.99	30.20
	MAE	4.43	4.76	3.95	3.07	3.87	12.87
	MAPE	5.41	5.61	6.37	6.61	6.69	13.01
USA male	RMSE	25.58	24.88	32.08	17.97	28.03	26.52
	MAE	16.40	16.04	17.84	10.96	15.09	13.44
	MAPE	11.38	11.49	12.78	7.19	9.90	9.94
USA female	RMSE	13.40	13.33	19.40	8.71	10.46	18.73
	MAE	8.49	8.45	10.77	5.43	6.19	9.57
	MAPE	8.32	8.32	8.14	6.41	6.50	8.37
UK male	RMSE	38.71	37.35	23.95	37.40	29.10	44.15
	MAE	20.01	19.84	13.31	20.35	15.30	22.25
	MAPE	10.56	10.78	7.47	9.00	8.76	15.90
UK female	RMSE	16.84	16.74	15.60	16.50	16.87	23.23
	MAE	8.90	8.89	8.49	8.90	8.84	11.17
	MAPE	8.46	8.50	6.99	7.47	6.56	9.18
Average	RMSE	20.47	20.30	19.23	16.80	20.76	28.00
	MAE	11.56	11.57	10.85	9.38	11.42	13.97
	MAPE	8.89	9.01	8.70	7.08	8.05	11.72

Note that RMSE and MAE are scaled to ($\times 10^{-4}$) and MAPE is a percentage.

Figure 6.3: $RMSE_{x,s}$, $MAE_{x,s}$ and $MAPE_{x,s}$ for years [2000, 2009] against ages [25, 84]

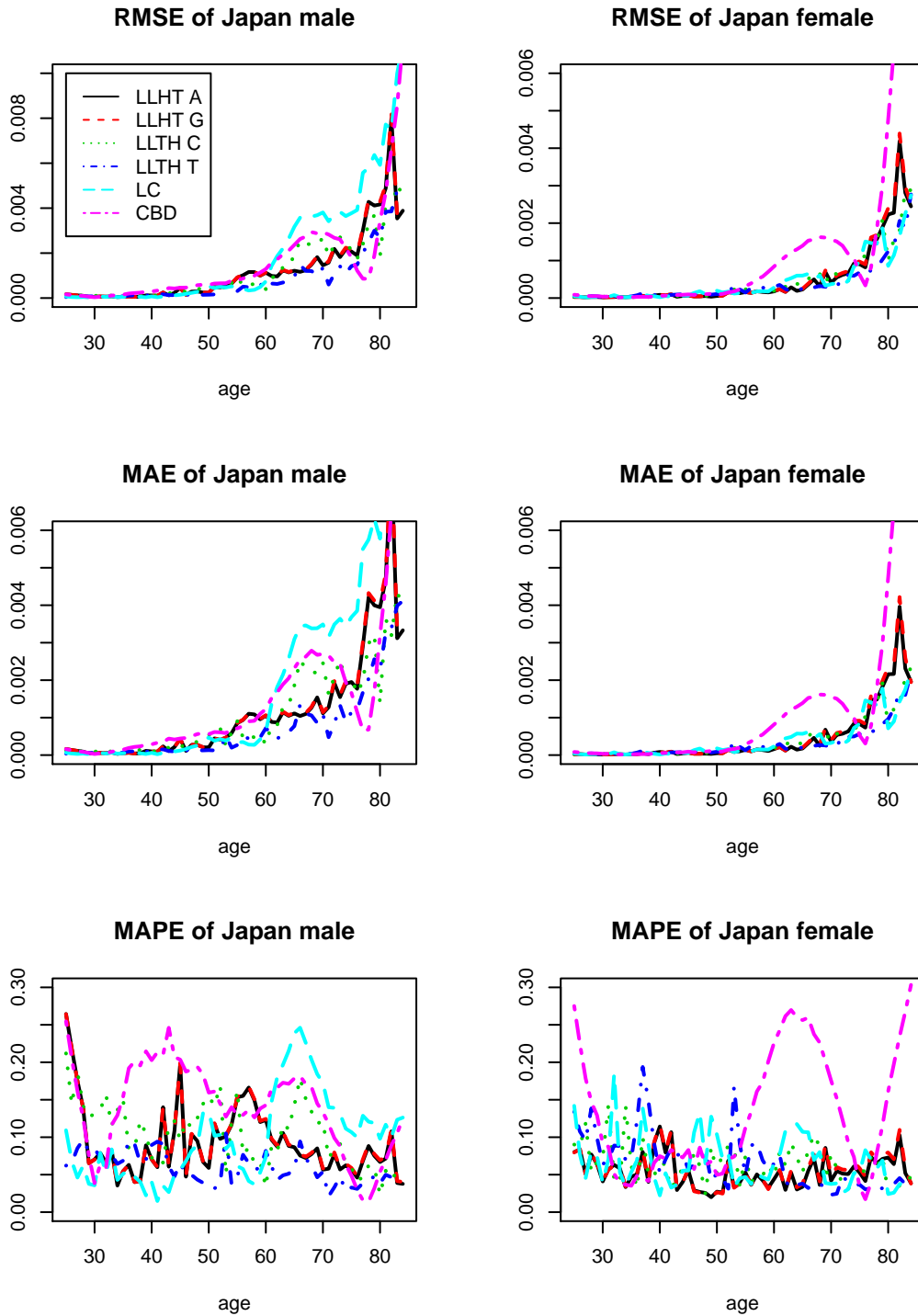


Table 6.3 lists the RMSEs, MAEs and MAPEs from the different mortality projection methods for Japan, the USA and the UK. The LLHT T method dominates in all three error statistics for Japanese males, and the LLHT C method also performs well in projecting this population. For the population of Japanese females, the LLHT T method wins in the RMSE and MAE, and loses slightly to the LLHT A, LLHT G and LLHT T methods in the MAPE. We note that smaller numbers in the RMSE and MAE but a bigger number in the MAPE usually indicate a relatively poorer forecasting performance at the younger ages. Looking at the cases of the USA, the LLHT T method beats all other methods in both the male and female populations, and the performance of the LLHT C method becomes the worst. By contrast, in the case of the UK, the LLHT C method is ranked the best in predicting the male and female populations. Comparing the averages over the six combinations of country and gender, the LLHT A and LLHT G methods can produce satisfactory figures though they are simple, and the LLHT T and CBD models perform the best and worst, respectively, in forecasting the deterministic mortality rates among the six projection methods.

To further explore the accuracy of the mortality projections, the $RMSE_{Ks}$, MAE_{Ks} and $MAPE_{Ks}$ for the age span [25, 84] against years [2000, 2009] for Japan are displayed in Figure 6.2. The $RMSE_{Ks}$, MAE_{Ks} and $MAPE_{Ks}$ from the LLHT A and LLHT G methods are close each other due to their approximate parameter pairs $(\hat{\alpha}_{x_L, x_U}^{tL, K}, \hat{\beta}_{x_L, x_U}^{tL, K})$. For the cases of $RMSE_K$ and MAE_K , the LLHT T method performs the best for both genders, and the Lee-Carter and CBD models perform the worst for male and female, respectively. For the case of $MAPE_K$, the CBD model produces the largest errors for both genders, and the LLHT T method yields the lowest errors for male and for years [2000, 2004] for female.

Figure 6.3 exhibits the $RMSE_{xs}$, MAE_{xs} and $MAPE_{xs}$ for the year span [2000, 2009] against ages [25, 84] for Japan. The $RMSE_{xs}$ and MAE_{xs} for the young ages are relatively smaller due to smaller scales. On the other hand, the $MAPE_{xs}$ are more volatile through ages. The phenomenon is mentioned above that even though the values of $RMSE_x$ and MAE_x have relatively small deviations due to very small rates at younger ages, the $MAPE_x$ is exaggerated because of a small denominator in (6.2). For the case of ages larger than 60 for Japanese males, the Lee-Carter model produces the largest $RMSE_{xs}$ and MAE_{xs} , causing their $RMSE$ and MAE to be the biggest (see Table 6.3). The same situation can be seen from the case of ages larger than 55 for Japanese females - the CBD model gives the poorest values. For the case of Japanese males, the $MAPE_{xs}$ from the LLHT T and

Lee-Carter models perform the best at young ages, but the $MAPE_x$ s from the Lee-Carter model have several peaks at the old ages. While for the case of Japanese females, all the projection methods produce relatively steady deviations except for the CBD model.

6.2 Illustrations for mortality projections in the stochastic view

The approaches to adding the stochastic characteristics to the deterministic LLHT-based methods are discussed in Chapter 4. To see the performances of the stochastic mortality projections, a 90% confidence interval is constructed on each of these mortality projections including those from the Lee-Carter and CBD models. The construction of the confidence intervals of the mortality rates by the LLHT-based methods is given by (4.4).

Both genders aged 30, 50 and 70 for Japan for illustrating the deterministic mortality projections are chosen as the representatives for the young, middle and old age groups in Figure 6.1. The same ages are also selected for the stochastic mortality projections and are presented in Figures 6.4, 6.5 and 6.6 for males and Figures 6.7, 6.8 and 6.9 for females. In the plots for the stochastic projections, the black solid curve is for the real mortality rates, and the shaded area represents the 90% confidence interval with the deterministic projection (the red dotted curve) in the middle of the shaded area. We set the same ranges for displaying all six stochastic mortality projections. The shaded areas and/or the red dotted curve do not show in some sub-figures because their shades/curves are out of range. Ideally, for a good stochastic mortality projection method, the confidence interval (the shaded area) should be as narrow as possible and should include the real mortality rates (the black solid line).

The Lee-Carter and CBD models have wider shaded areas than the LLHT-based models do; in some cases, the wider confidence interval captures more of the real mortality rates, but it also create much larger uncertainty in the projected mortality rates. Sometimes even with a wider confidence interval, the Lee-Carter and CBD models still do not cover any of the real mortality rates. The LLHT T method has a pretty good prediction for males aged 30, 50 and 70; even though its confidence interval is tighter than others, the shaded area covers a lot of the real mortality rates. For the case of females, the performances of the LLHT A and LLHT G methods are satisfactory. Most or all of the real mortality rates are

covered by the confidence intervals from the Lee-Carter model for ages 30 and 70 and from the CBD model for ages 30 and 50. The less wide confidence interval for age 70 from the LLHT T method still covers most of the real mortality rates.

Table 6.4 gives the number of the real mortality rates falling into the shaded areas from the six combinations of gender and age, which is a simple measure of forecasting performances of the stochastic projections. Though the Lee-Carter and CBD models have similar coverage, they both also have far wider confidence intervals than the LLHT-based models. Considering Tables 6.4 (based on six combinations of gender and age for Japan) and 6.3 (based on a wider age span and six combinations of country and gender), we conclude that the LLHT T method is ranked the best for the performance of forecasting the mortality rates.

Table 6.4: The number of the real mortality rates included in the 90% confidence intervals

Counts	LLHT A	LLHT G	LLHT C	LLHT T	LC	CBD
Male 30	2	2	0	3	9	8
Male 50	2	2	0	7	0	0
Male 70	3	3	0	5	0	0
Female 30	6	6	2	3	5	7
Female 50	4	4	1	4	0	8
Female 70	4	3	0	7	10	0
Total	21	20	3	29	24	23
Average	3.50	3.33	0.50	4.83	4	3.83

6.3 Evaluation of mortality-linked security

Besides the comparison of the different mortality projection methods for their performances of forecasting mortality rates, this project also studies how both the life insurer and annuity provider are affected by some mortality-linked securities based on the projected deterministic and stochastic mortality rates from the underlying mortality models.

The gain/loss at the issue for a life insurer without a mortality-linked security would be the premium income from the NSP at issue minus the present value of the realized death

Figure 6.4: 90% confidence intervals for Japan male aged 30

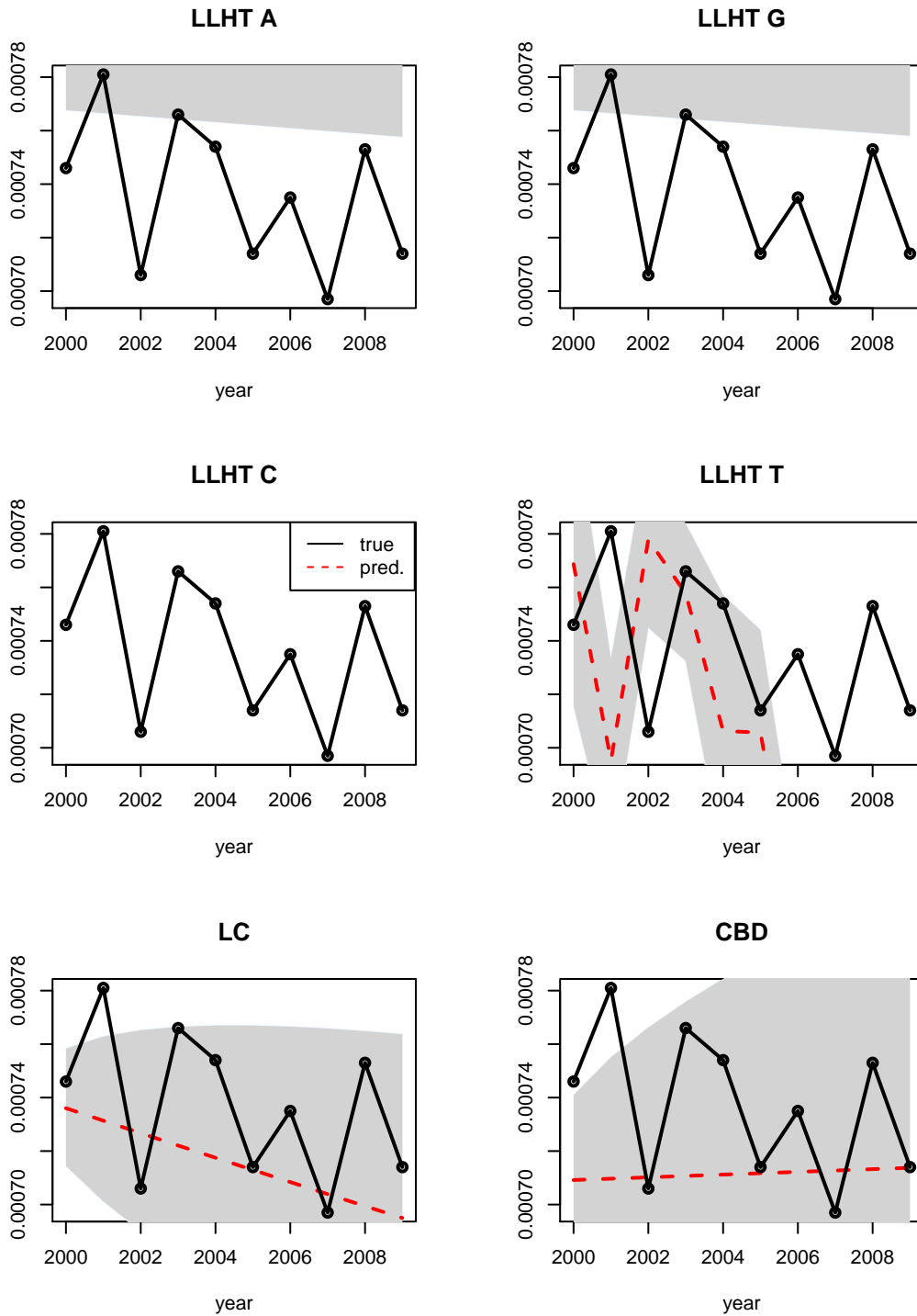


Figure 6.5: 90% confidence intervals for Japan male aged 50

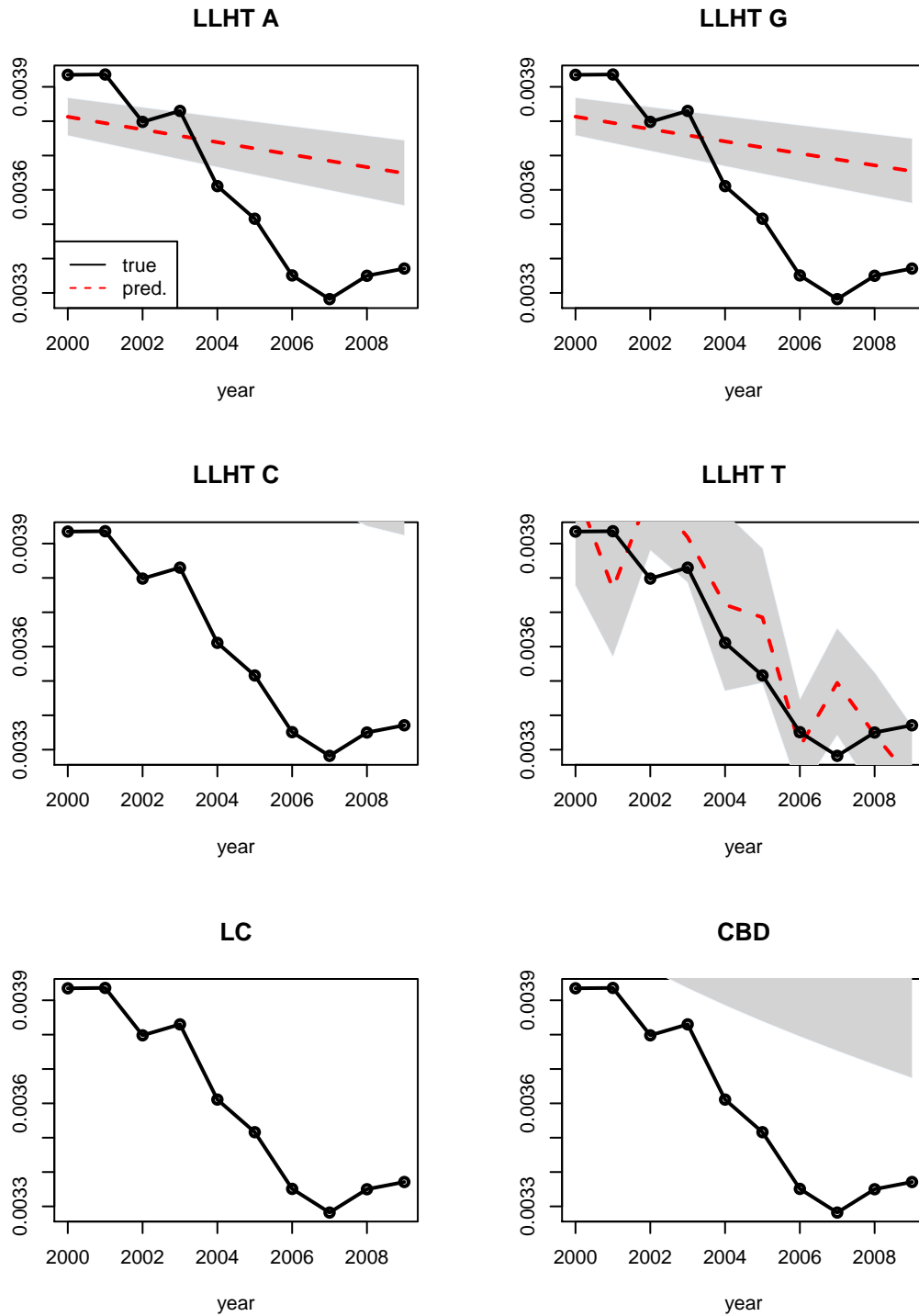


Figure 6.7: 90% confidence intervals for Japan female aged 30

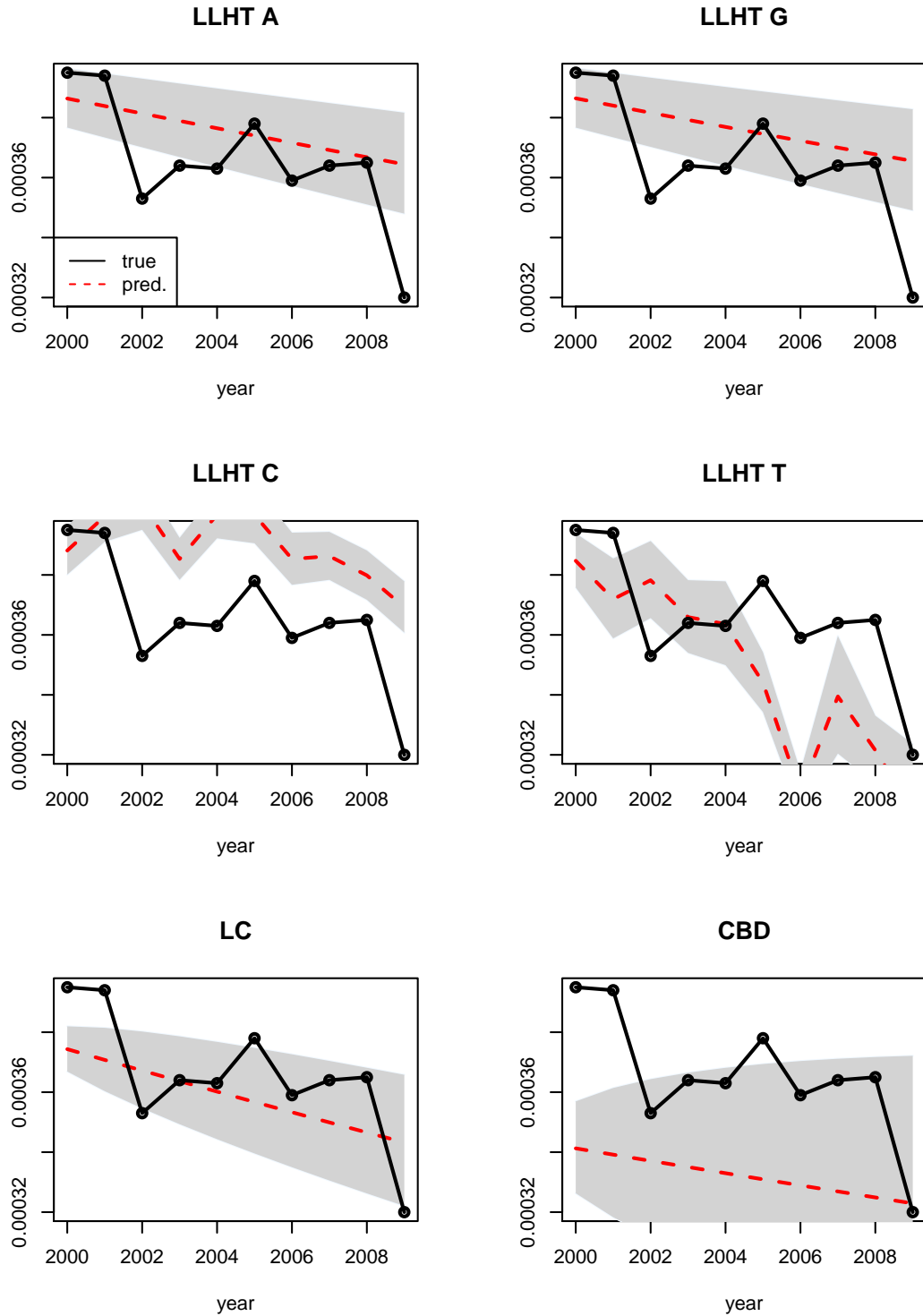


Figure 6.8: 90% confidence intervals for Japan female aged 50

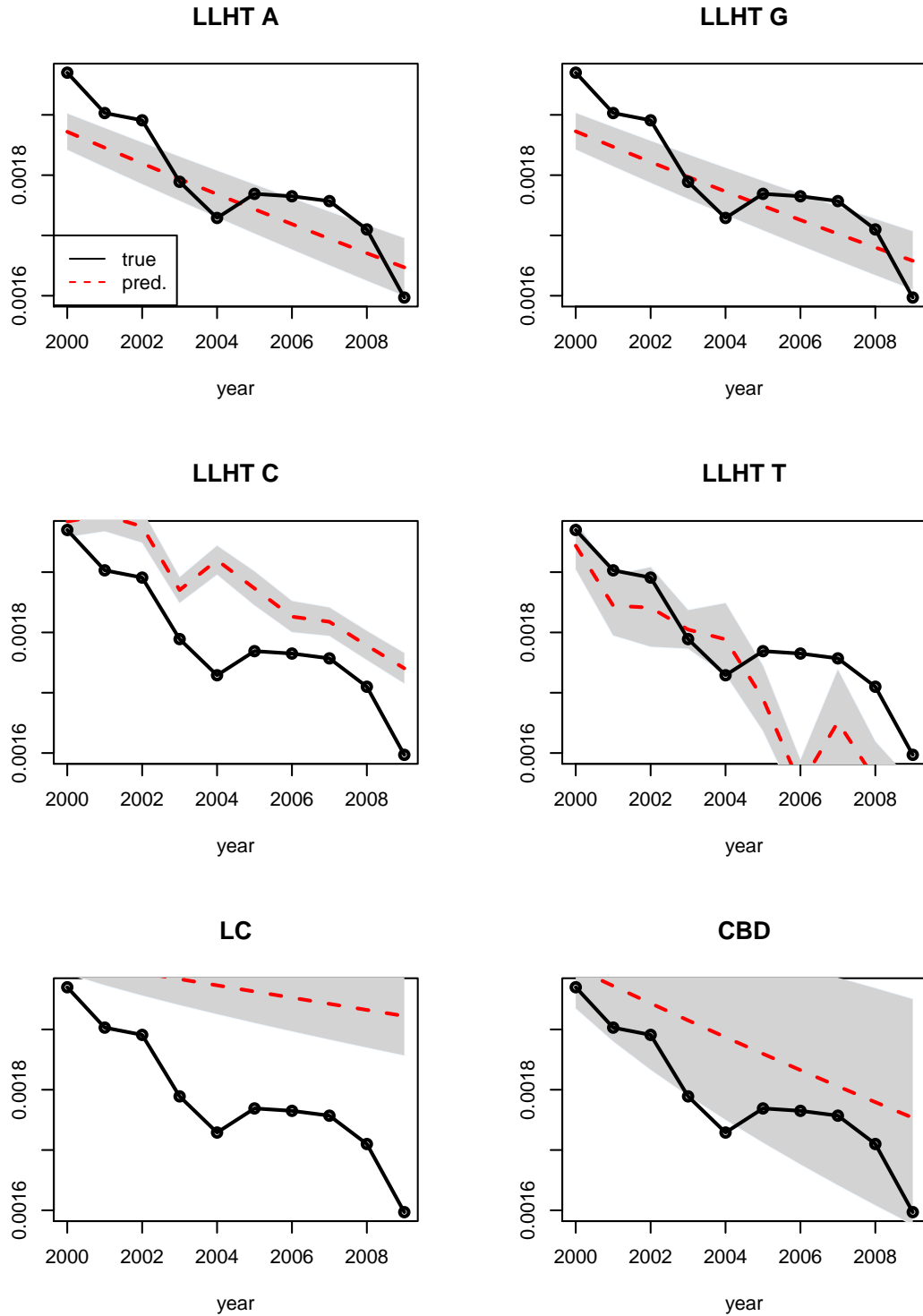
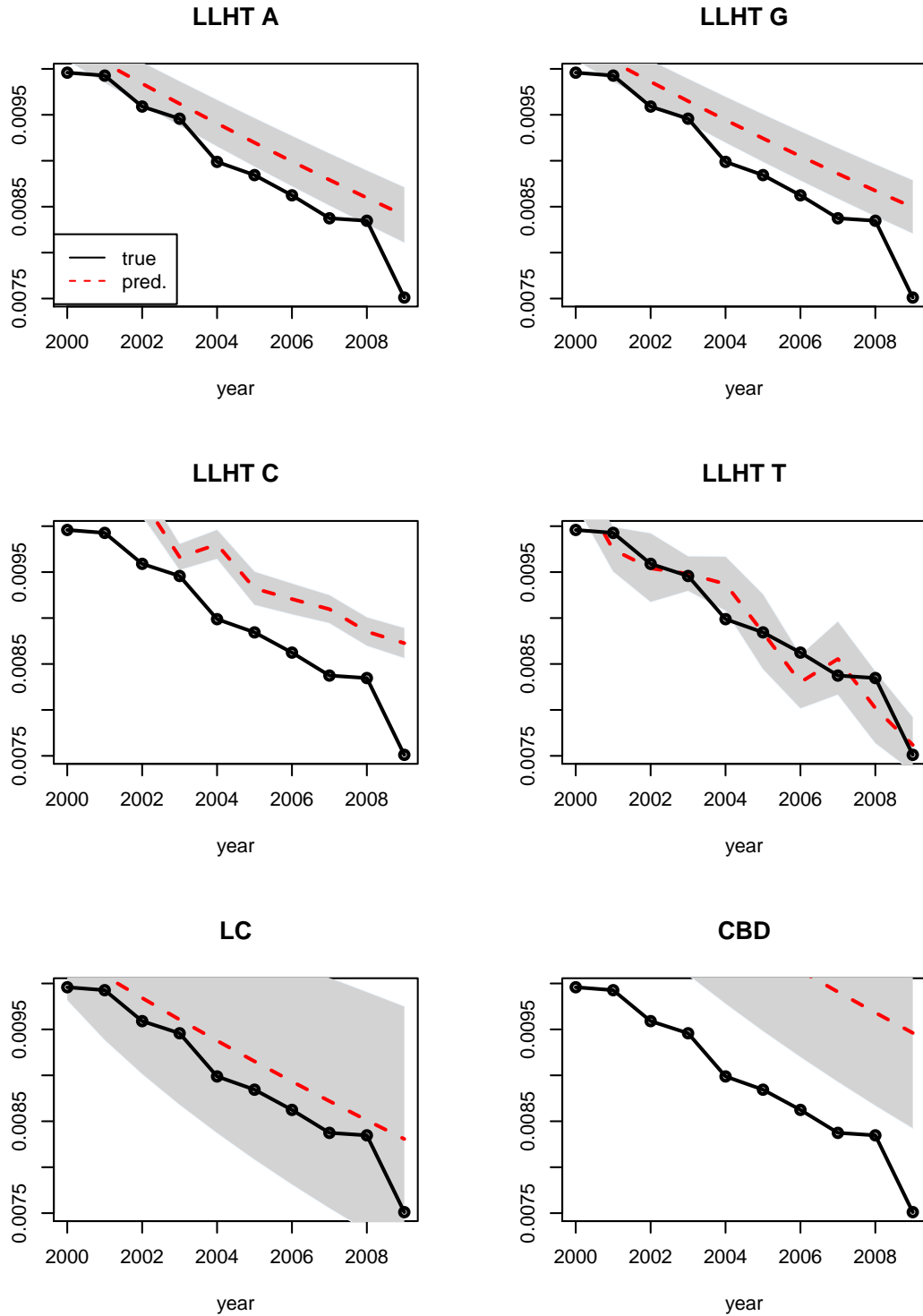


Figure 6.9: 90% confidence intervals for Japan female aged 70



benefit payout. With a mortality-linked security involved, the cash flow from the mortality-linked security needs to be included to calculate the net gain/loss. The following are the gain/loss ratios for the life insurer and the annuity provider, respectively:

$$[(A_{x:\overline{n}}^1)^e - (A_{x:\overline{n}}^1)^r + NCF_A]/(A_{x:\overline{n}}^1)^e \quad (6.3)$$

and

$$[(a_{x:\overline{n}})^e - (a_{x:\overline{n}})^r + NCF_a]/(a_{x:\overline{n}})^e, \quad (6.4)$$

where $A_{x:\overline{n}}^1$ stands for the NSP of the discrete n -year term life insurance for an individual aged x , and $a_{x:\overline{n}}$ is the NSP of the n -year temporary annuity-immediate, and NCF_A and NCF_a represent the net cash flows from a mortality-linked security for the life insurer and the annuity provider, respectively. The 'expected' and 'realized' values are abbreviated by 'e' and 'r', respectively. Usually, the net cash flow (NCF_A or NCF_a) contains the incoming cash flow from the SPV for the realized mortality rate less the cost of the mortality-linked security, which is explained in detail in what follows.

Consider a discrete 10-year term life insurance with a death benefit of 1 issued to insureds aged 40 by a life insurer and a 10-year temporary annuity-immediate with annual payments of 1 issued to insureds aged 55 by an annuity provider. The mortality rates for the buyers of life insurance are usually higher than the ones for the buyers of annuities. Hence, male and female mortality rates are chosen for the life insurance and annuities, respectively. The projected cohort mortality rates are adopted for this study. In other words, for an individual aged x at issue time t , the mortality rates for a period of 10 years are $q_{x+k,t+k}, k = 0, \dots, 9$. With relatively good performance in the figures of Table 6.3, the Lee-Carter, LLHT C and LLHT T methods are selected for studying of the mortality-linked securities discussed. Figure 6.10 presents the projected cohort mortality rates for Japanese males aged 40 and Japanese females aged 55 used for pricing insurance and annuity policies, respectively, with issue year $t = 2000$. At time of issue, the real mortality rates are unknown until the end of year 10. The deterministic mortality rates from the Lee-Carter, LLHT C and LLTH T models are projected and used for pricing the life insurance, annuity and mortality-linked securities. We observe from Figure 6.10 that the LLHT T model underestimates the future mortality rates while the Lee-Carter and LLHT C models overestimate those for both Japanese males aged 40 and females aged 55. Then 500,000 simulated mortality rate paths from each of the stochastic mortality projection methods are obtained with the

same generated random numbers to calculate actual death benefit and annuity payments as well as the cash flows for the mortality-linked securities. Here we assume 3% for both the expected and realized rates of return.

6.3.1 q-forward

Purchasing mortality-linked securities is one of the approaches to hedging the mortality/longevity risks for both the life insurer and annuity provider which would like to participate in the q-forward at each of the policy anniversaries with a certain face amount F , $F < 1$. This means that there would be exchanges of mortality rates at the end of years 2000, 2001, ..., 2009. Due to the absence of LifeMetrics Index for Japan, the mortality rates for Japan from the Human Mortality Database serve as the reference mortality rates for the q-forward.

Table 6.5 gives the gain/loss ratios for the life insurer and annuity provider with various face amount F based on (6.3) and (6.4), respectively. For the q-forward between J.P. Morgan and the life insurer, NCF_A in (6.3) will be the sum of the settlement values calculated by (5.1) and discounted to the current date, less the risk premium calculated by (5.4) for $T = 1, \dots, n$. For the q-forward between J.P. Morgan and the annuity provider, NCF_a in (6.4) is the sum of the settlement values calculated by (5.2) and discounted to the current date, plus the risk premium calculated by (5.4) for $T = 1, \dots, n$. Mathematically, the net cash flow for the life insurer and annuity issuer in one simulated mortality path $\{q_{x+T,T}^r : T = 1, \dots, n\}$ are

$$NCF_A = F \cdot \sum_{T=1}^n \left[(q_{x+T,T}^r - q_{x+T,T}^f) \cdot v^T - (q_{x+T,T}^e - q_{x+T,T}^f) \cdot v^T \right] \quad (6.5)$$

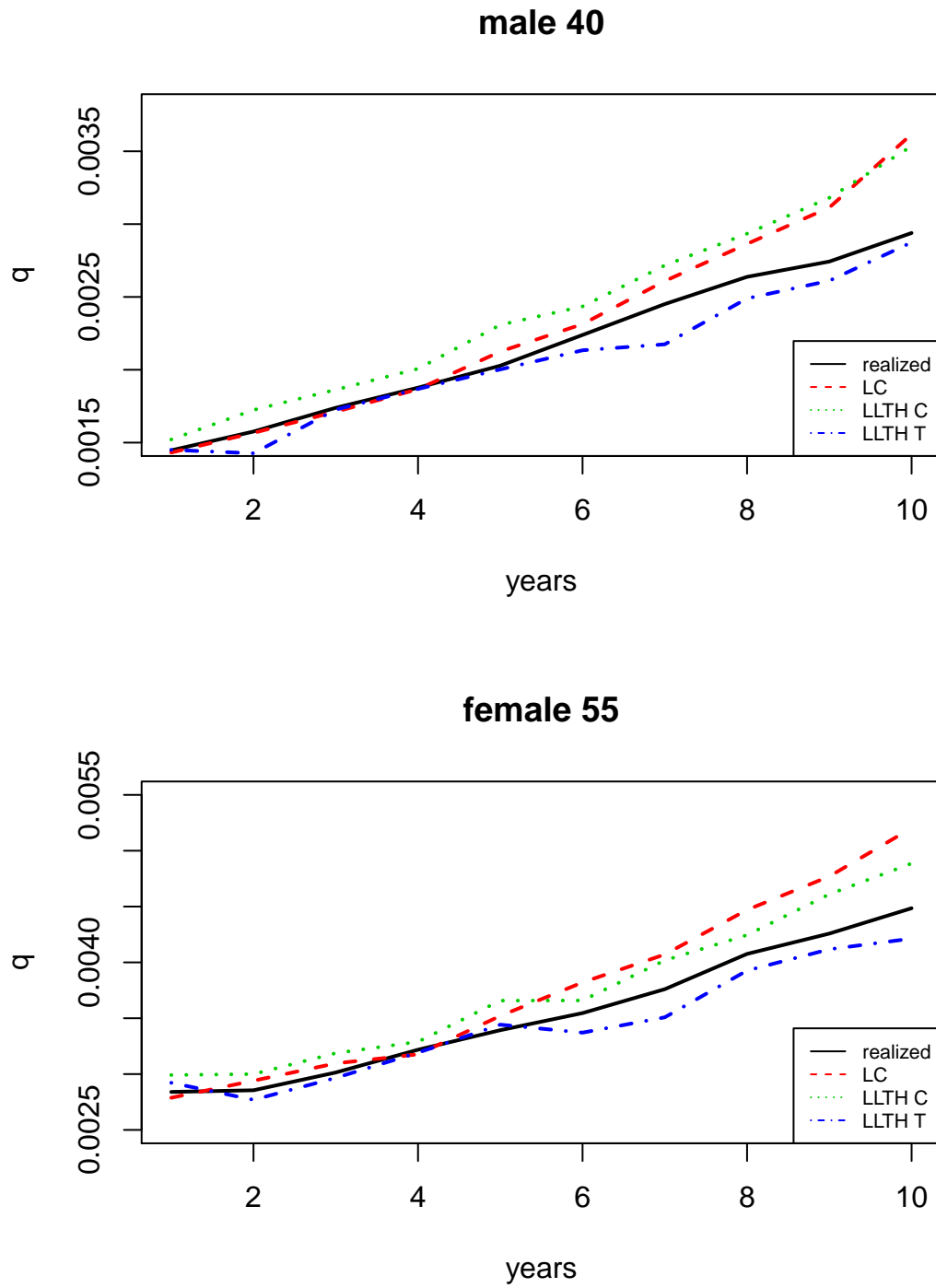
and

$$NCF_a = F \cdot \sum_{T=1}^n \left[(q_{x+T,T}^f - q_{x+T,T}^r) \cdot v^T + (q_{x+T,T}^e - q_{x+T,T}^f) \cdot v^T \right], \quad (6.6)$$

where the discount rate for pricing is the same as that for realization, and n is the term of the life insurance. Note that $F \cdot (q_{x+T,T}^r - q_{x+T,T}^f) \cdot v^T$ and $F \cdot (q_{x+T,T}^f - q_{x+T,T}^r) \cdot v^T$ in (6.5) and (6.6) represent the present values of the settlement value paid at year T for the life insurer and annuity provider, respectively, and $F \cdot (q_{x+T,T}^e - q_{x+T,T}^f) \cdot v^T$ in (6.5) and (6.6) is the risk premium for year T .

For each fixed face amount, the LLHT T method produces the smallest gain/loss ratios

Figure 6.10: The real and projected mortality rates for Japan in year 2000



in the absolute value, while the Lee-Carter model yields the largest ones, which means that the LLHT T method can give the most accurate estimates for the gain/loss ratios for both the life and annuity policies with the q -forward.

Table 6.5: Gain/loss ratios (0.01%) for the life insurer and the annuity provider

	insurance for Japan male aged 40			annuity for Japan female aged 55		
F	Lee-Carter	LLHT C	LLHT T	Lee-Carter	LLHT C	LLHT T
0.0	-2.5125	-0.0478	-0.0076	0.0130	0.0006	0.0001
0.1	-2.2562	-0.0429	-0.0068	0.0126	0.0006	0.0001
0.2	-1.9999	-0.0381	-0.0061	0.0122	0.0006	0.0001
0.3	-1.7436	-0.0333	-0.0053	0.0119	0.0006	0.0001
0.4	-1.4873	-0.0284	-0.0045	0.0115	0.0006	0.0001
0.5	-1.2311	-0.0236	-0.0038	0.0111	0.0006	0.0001
0.6	-0.9748	-0.0188	-0.0030	0.0107	0.0005	0
0.7	-0.7185	-0.0140	-0.0022	0.0103	0.0005	0
0.8	-0.4622	-0.0091	-0.0015	0.0099	0.0005	0
0.9	-0.2059	-0.0043	-0.0007	0.0095	0.0005	0
1.0	0.0503	0.0005	0.0001	0.0091	0.0005	0

For the q -forward between the life insurer and J.P. Morgan, the negative values in the 2nd, 3rd and 4th columns of Table 6.5 for the case of zero face amount ($F = 0$) come from the fact that the average of $(A_{x:\overline{n}|}^1)^r$ s based on the simulated mortality rates is slightly higher than $(A_{x:\overline{n}|}^1)^e$ with the deterministic mortality rates (setting $NCF_A = 0$ in (6.3)), from which it can be inferred that the average of the simulated mortality rates is higher than the deterministic mortality rate ($\bar{q}_x^r > q_x^e$). Since the fixed mortality rate in the q -forward contract is designed to be set lower than the "best estimate" mortality rate or the deterministic mortality rate ($q_x^e > q_x^f$ by (5.3)), the average of the simulated mortality rates is definitely higher than the fixed mortality rate ($\bar{q}_x^r > q_x^f$), making the life insurer get a positive settlement ($F \times (\bar{q}_x^r - q_x^f)$) at maturity. This is the reason why the values in the 2nd, 3rd and 4th columns of Table 6.5 increase with the face amount F .

The 5th, 6th and 7th columns in Table 6.5 are for the q -forward between the annuity provider and J.P. Morgan. The positive values for $F = 0$ are a result of $(a_{x:\overline{n}|})^e$ based on the deterministic mortality rates being larger than the average of $(a_{x:\overline{n}|})^r$ s based on the simulated mortality rates, setting $NCF_a = 0$ in (6.4). In this case, the annuity provider will

experience smaller annuity payments than expected in the annuity policy, which results in a mortality gain for the annuity provider. This good news for the annuity provider leads to the deficient settlement in the q -forward contract because the annuity provider in the q -forward is the recipient of the fixed mortality rate and the payer of the floating mortality rate. Moreover, since $\bar{q}_x^r > q_x^f$, the settlement for the annuity provider is $F \times (q_x^f - \bar{q}_x^r) < 0$ which makes the values in the 5th, 6th and 7th columns of Table 6.5 decrease with the face amount F .

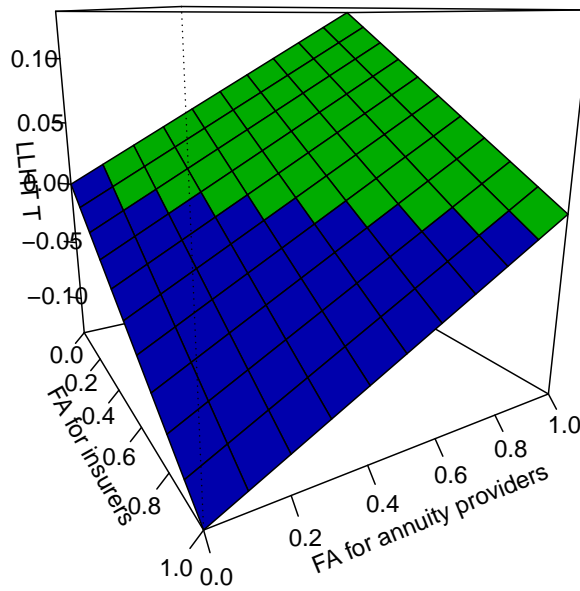
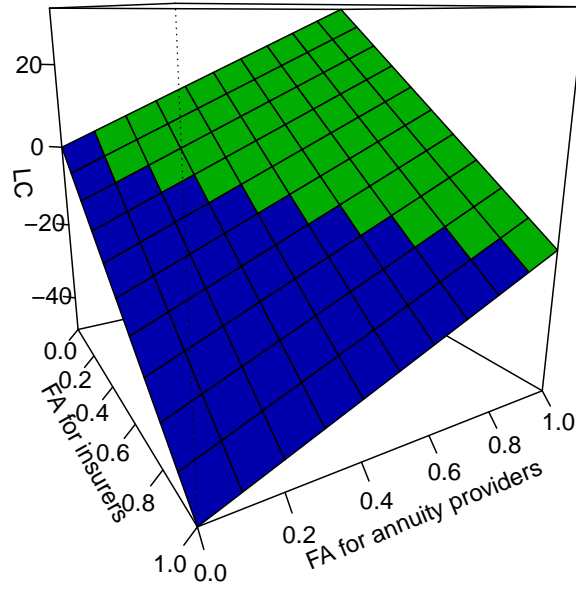
The net income, $-NCF_a - NCF_A$, for the SPV (J.P. Morgan) generated by participating in the q -forward business with both the life insurer and the annuity provider should also be evaluated to see if the SPV will make profit. The net incomes constructed with the combinations of different face amounts (FA) for the life insurer and the annuity provider are shown in Figure 6.11. The net incomes differ with the projection methods; the net incomes from the Lee-Carter model are more volatile than those from the LLHT T method. The maximum gain and the maximum loss are approximately equal in absolute value for the LLHT T model, while the maximum loss is almost double the maximum gain in absolute value for the Lee-Carter model.

6.3.2 Modified q -forward with tranches

With the tranche defined and the tranche rates specified in the previous Chapter, both the life insurer and the annuity provider purchase the modified q -forward to hedge the mortality/longevity risks at each of the policy anniversaries. The gain/loss ratios calculated by (6.3) and (6.4) for the life insurer and the annuity provider, respectively, are presented in Table 6.6. Each of NCF_A and NCF_a is the sum of the settlement amounts calculated by (5.5) for all applicable tranches and discounted to current date, less the premiums calculated by (5.6) for $t = 1, \dots, n$.

It is seen again that $\bar{q}_x^r > q_x^e$ (the average of the simulated mortality rates is higher than the deterministic mortality rate) from the row with the face amount $F = 0$ in Table 6.6, which leads to negative and positive gain/loss ratios for the life insurer and the annuity provider, respectively, similar to those in Table 6.5. The gain/loss ratios from both of the mortality projection methods for the annuity provider are increasing with the face amount, while those for the life insurer are decreasing. Again, the LLHT T method produces lower

Figure 6.11: Net incomes ($\times 10^{-6}$) for J.P. Morgan with the q-forward business



gain/loss ratios in absolute value than the Lee-Carter model

Table 6.7 presents the premium of the modified q-forward and the mean of the settlement amount with 500,000 simulated paths. The simulation results and the expectations of the future payments have a big disparity for the modified q-forward. The premium for the life insurer is greater than the settlement amount, while the premium for the annuity provider is lower than the settlement, which can explain why the gain/loss ratios decrease for the life insurer and increase for the annuity provider. Moreover, for the annuity provider, the difference between the premium and settlement in the absolute value for the Lee-Carter model is larger than that for the LLHT T method (64.21×10^{-6} versus 37.21×10^{-6}), whereas for the life insurer, the situation reverses (15.79×10^{-6} versus 41.67×10^{-6})

Table 6.6: Gain/loss ratios (%) for the life insurer and the annuity provider

	Insurance for Japan male aged 40		Annuity for Japan female aged 55	
F	Lee-Carter	LLHT T	Lee-Carter	LLHT T
0	-3.512	-0.473	0.032	0.007
0.1	-3.594	-0.681	0.033	0.008
0.2	-3.677	-0.889	0.034	0.008
0.3	-3.759	-1.097	0.035	0.008
0.4	-3.842	-1.306	0.035	0.009
0.5	-3.924	-1.514	0.036	0.009
0.6	-4.007	-1.722	0.037	0.010
0.7	-4.089	-1.930	0.038	0.010
0.8	-4.172	-2.139	0.038	0.011
0.9	-4.254	-2.347	0.039	0.011
1	-4.337	-2.555	0.040	0.012

Table 6.7: premium and settlement amount ($\times 10^{-6}$) of the modified q-forward at issue

	Insurance for Japan male aged 40		Annuity for Japan female aged 55	
	Lee-Carter	LLHT T	Lee-Carter	LLHT T
settlement	25.59	13.29	90.31	50.81
premium	41.38	54.96	26.10	13.60
difference	-15.79	-41.67	64.21	37.21

Chapter 7

Conclusion

The linear hazard transform (LHT) model proposed by Tsai and Jiang (2010) and the linear logarithm hazard transform (LLHT) model by Tsai (2012) are simple to understand and implement.

Regarding the projection approaches based on the LLHT model, in addition to the arithmetic and geometric projection methods by Tsai and Jiang (2010), two new approaches are introduced in this project. Accuracy of the projected mortality rates depends on the selected mortality rates (the age span, the year span, the country, the gender, etc.). Empirical mortality data from both genders of the USA, the UK and Japan show that the LLHT T method is overall the best among the six models/methods explored and the LLHT C method is better than the CBD model based on the illustrated figures of the MAPE, MAE and RMSE.

When mortality data are only available for two years, the Lee-Carter, CBD, LLHT C and LLHT T models do not work. Another disadvantage for the LLHT C and LLHT T models is that they can only project mortality rates up to some number of years, while the Lee-Carter and CBD do not have this problem. The LLHT A and LLHT G models are able to project mortality rates as many years into the future as one likes with the required mortality data as few as two years.

The characteristics of the proposed stochastic mortality models based on the LLHT model give us the ability to price mortality-linked securities through simulations. Numerical examples for q-forward and modified q-forward show that the LLHT T model gives the most accurate results and the Lee-Carter model produces the least ones among three models.

In conclusion, the LLHT-based models are not only simpler to understand and implement with fewer parameters than the Lee-Carter and CBD models but also produce smaller projection errors for most cases. The disadvantage of the LLHT C and LLHT T models is that they can only project mortality rates up to some number of years. Creating some variations of the LLHT model that resolve this shortcoming is the topic of the further research.

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