MODELING DEPENDENCE INDUCED BY A COMMON RANDOM EFFECT AND RISK MEASURES WITH INSURANCE APPLICATIONS

by

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Abstract

Random effects models are of particular importance in modeling heterogeneity. A commonly used random effects model for multivariate survival analysis is the frailty model. In this project, a special frailty model with an Archimedean dependence structure is used to model dependent risks. This modeling approach allows the construction of multivariate distributions through a copula with univariate marginal distributions as parameters. Copulas are constructed by modeling distribution functions and survival functions, respectively. Measures of the dependence are applied for the copula model selections. Tail-based risk measures for the functions of two dependent variables are investigated for particular interest. The statistical application of the copula modeling approach to an insurance data set is discussed where losses and loss adjustment expenses data are used. Insurance applications based on the fitted model are illustrated.

Keywords: Multivariate distribution; Copula; Common random effects; Measure of dependence; Measures of tail dependency; Risk measures; VaR; CTE

To my family.

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Chapter 1

Introduction

Many financial and actuarial problems involve more than one random variable. Interactions of these random events are of particular interest to practitioners. For example, an investor needs to consider the returns of all securities included in the same investment portfolio. A life insurer may be interested in the mortality of multiple lives insured under the same policy. Property and casualty insurers are concerned about the losses from different lines of business.

Although it is convenient to assume that the relevant random variables are independent, this is often inappropriate. For instance, securities have a tendency to move together in the same direction because of the market risk that stems from economy-wide factors affecting all securities. Life insurance policies sold to married couples involve dependent risks (spouses' remaining life times). Catastrophe insurance has to deal with the consequences resulting from perils such as hurricanes, earthquakes, or tornadoes. It is of great importance to study the relationships among different dimensions of an outcome and model the dependence structure of these random variables.

Furthermore, failure to take proper account of extremal behavior in the tails may result in devastating consequences. Large amounts of insurance losses have significant impact on the solvency of insurers or reinsurers. Also, abnormal movements in interest rates or stock prices can dramatically affect the values of assets or liabilities of financial institutions. As a result, it is also necessary to capture the dependence of random variables at extreme values.

1.1 Background and Motivation

The normal distribution has long dominated the study of multivariate distributions, based on the fact that many elements of nature follow normal distributions and are related to other normally distributed variables. As far as applications are concerned, multivariate normal distributions are appealing because the marginal distributions are also normal and the association between two normal variables can be described by their marginal distributions and the correlation coefficient. The modern portfolio theory, which is based on the assumption of multivariate normal returns, establishes the variance (or standard deviation) as a risk measure and the correlation coefficient between returns as a measure of dependence (Markowitz, 1952, 1959). The drawback of models based on the normal distribution is that they cannot capture the extremal behavior in the tails. The distribution of financial asset returns is leptokurtic, which means the tails are fatter than those implied by normal distributions. Financial asset returns also tend to be negatively skewed. Both of these suggest that the multivariate normal model for financial assets is likely to understate the actual risk.

Copulas provide a convenient way to study the dependence between random variables. According to Sklar's Theorem, any joint distribution can be expressed in terms of a copula. A copula separates the joint distribution into two components – the marginal distribution of individual variables, and dependence parameter(s) that capture(s) the interdependence of the marginal distributions. That is, a copula expresses the joint distribution of random variables as a function of the marginal distributions of each variable.

This project aims to model the joint behavior of random variables, with an emphasis on tail dependency. Models with common random effects are used to study the joint behavior of random variables. Copulas are derived from modeling distribution functions and survival functions, respectively. Tail-based risk measures for functions of dependent risks are derived. The statistical applications of the copula modeling approach to insurance data are discussed where losses and loss adjustment expenses data are used. Insurance applications based on the fitted model are illustrated.

1.2 Literature Review

Copulas have been extensively studied in recent years. Joe (1997) and Nelsen (2006) gave comprehensive discussions of copula functions and their statistical properties. Trivedi and

Zimmer (2005) provided a guide to copula modeling, with special attention dedicated to issues related to estimation and mis-specification. In Chapter 2, we will present the basic statistical properties of multivariate distributions and copulas that are useful for our study in this project.

This literature review focuses on multivariate modeling and its connection with copulas, and results from previous research that applied copula models to actuarial and financial problems. There is an extensive literature on copulas and multivariate models, and as a result, we only provide a review of those that are most relevant to our random effects models studied in Chapter 3, measures of dependence and tail dependency presented in Chapter 4, and tail-based risk measures discussed in Chapter 5. This review is not meant to give a complete list of all related research.

Random effects models are of particular importance in modeling heterogeneity. A widely used random effects model in multivariate survival analysis is the frailty model introduced by Vaupel et al. (1979).

Oakes (1989, 1994) considered bivariate and multivariate survival models induced by frailties. We start our literature review by presenting the frailty models first. For a continuous random survival time, T, the survival function is defined as

$$S(t) = P(T > t) = 1 - F(t) = 1 - \int_0^t f(s)ds,$$

where $F(t)$ and $f(t)$ are the distribution function and density function of T, respectively. The hazard function $h(t)$ can be derived as

$$h(t) = -\frac{\partial \ln S(t)}{\partial t} = \frac{f(t)}{S(t)}.$$

Explanatory variables Z can be incorporated into survival analysis using Cox's proportional hazards model (Cox, 1972), in which the hazard function is represented as

$$h(t, Z) = e^{\beta Z} b(t),$$

where $b(t)$ is the baseline hazard function, and β is a vector of regression parameters.

Let $\gamma = e^{\beta Z}$. Integrating and exponentiating the negative hazard, Cox's proportional hazards model can then be expressed as

$$S(t|\gamma) = e^{-\int_0^t h(s,Z)ds} = B(t)^{\gamma},$$

where $B(t) = e^{-\int_0^t b(s)ds}$ is the survival function corresponding to the baseline hazard function. Parameter γ is called a frailty in the sense that a larger value of γ implies a smaller survival probability $S(t|\gamma)$, indicating poorer survival.

Oakes (1989, 1994) then illustrated how the dependency among multiple survival times can be modeled with frailties. Assuming that two survival times T_1 and T_2 are independent given the frailty γ, we have

$$\begin{aligned} P(T_1 > t_1, T_2 > t_2 \,|\, \gamma) &= P(T_1 > t_1 \,|\, \gamma)\, P(T_2 > t_2 \,|\, \gamma) \\ &= S_1(t_1 \,|\, \gamma)\, S_2(t_2 \,|\, \gamma) \\ &= B_1(t_1)^{\gamma}\, B_2(t_2)^{\gamma}. \end{aligned}$$

Taking expectations over the potential values of γ, a realization of random variable Γ, we can get the following joint multivariate survival function,

$$P(T_1 > t_1, T_2 > t_2) = E_{\Gamma}\left[B_1(t_1)\, B_2(t_2)\right]^{\Gamma}.$$

That is, multivariate survival models result when some unknown factors induce dependence between random variables.

Marshall and Olkin (1988) proposed an approach of generating multivariate distributions by mixtures. The mixture model introduced by Marshall and Olkin (1988) takes the following form,

$$F(x) = \int H(x)^{\theta} dG(\theta),$$

where H and G are univariate distribution functions, and $\theta > 0$.

Let φ be the Laplace transform of G. Then $F(x) = \varphi(-\ln H(x))$. As a result, the univariate distribution function H can be expressed as

$$H(x) = e^{-\varphi^{-1}(F(x))}.$$

Marshall and Olkin (1988) also considered the bivariate mixture model given by

$$F(x_1, x_2) = \int\int H_1(x_1)^{\theta_1} H_2(x_2)^{\theta_2} dG(\theta_1, \theta_2). \tag{1.1}$$

Denote the marginal distributions of G by G_1 and G_2. The marginal distribution functions of $F(x_1, x_2)$ are given by

$$F_i(x) = \int H_i(x)^{\theta_i} dG_i(\theta_i), \quad i = 1, 2.$$

It follows that if

$$H_i(x) = e^{-\varphi_i^{-1}(F_i(x))},$$

where φ_i is the Laplace transform of $G_i, i = 1, 2$, then F given by (1.1) is a bivariate distribution function with marginal distributions F_1 and F_2 as parameters.

Models of common random effects that we will introduce in Chapter 3 have similar flavor. We will expand the above mentioned models by Oakes (1989, 1994) and Marshall and Olkin (1988), and use common random effects to model the dependence between random variables. A unified approach will be applied to models based on distribution functions and survival functions, respectively. Different bivariate distributions and their associated copulas are resulted by modeling distribution functions and survival functions. A variety of distributions for the common random effect will be discussed.

Frees and Valdez (1998) introduced actuaries to the concept of copulas, and illustrated how the frailty models proposed by Oakes (1989, 1994) can be applied to actuarial science, including estimation of joint life mortality and dependent decrement models. Frees and Valdez (1998) also showed how to simulate and fit copulas, and discussed the usefulness of copula functions by pricing a reinsurance contract and estimating expenses for pre-specified losses.

Dupuis and Jones (2006) illustrated the usefulness of multivariate extreme value theory and its actuarial applications. They used copula models and theoretical results from extreme value theory to study the extremal behavior of the joint distribution of random variables, with special attention dedicated to the asymptotic behavior of the dependence structure at extreme values. Venter (2002) also emphasized the correlation among large losses, i.e., in the right tails of the loss distributions. Various aspects of copulas regarding dependence structure and tail dependency were discussed in both Dupuis and Jones (2006) and Venter (2002). We will review the measures of dependence and tail dependency in Chapter 4.

Copula models have also been applied to other areas of actuarial research such as classical risk theory. Albrecher et al. (2011) considered dependent risks in the setting of classical risk theory. They modeled the dependency among claim sizes and among claim inter-occurrence times with copulas, and derived explicit formulas for ruin probabilities.

Extensive applications of copulas can be found in finance literature. Monograph by Cherubini et al. (2004) was dedicated to the financial applications of copula models, including simulations of market scenarios, credit risk applications, and options pricing. However, the arguably most influential work of the financial application of copula was Li (2000), which has been quoted numerous times in academia, used (or abused) by investment managers on Wall Street, and mentioned by news and media. Li (2000) introduced a copula model in finance to calibrate defaults. A random variable called the time-until-default was used to characterize the default, and the use of normal copulas was illustrated in the valuation of credit derivatives, such as credit default swaps and first-to-default contracts. Some even "blamed" the work of Li (2000) for the global financial crisis in 2008[1].

Numerous risk measures have been proposed in financial and actuarial literature. Denuit et al. (2005) provided a detailed overview of risk measures, their respective properties, and theories behind different measures of risk. Jorion (2007) focused on the value-at-risk (VaR), an extensively used risk measure in finance, and illustrated the use of VaR for integrated risk management. A review of tail-based risk measures will be given in Chapter 5 before they are applied to our model of common random effects.

The rest of this project is organized as follows. A brief review of multivariate distributions and copulas and their statistical properties is given in Chapter 2. In Chapter 3, models with common random effects are used to study the joint behavior of two random variables. Copula models are derived from modeling distribution functions and survival functions. Chapter 4 is dedicated to measures of dependence and tail dependency, and the use of tail dependency measures for copula model selections. Chapter 5 presents tail-based risk measures for functions of dependent risks. In Chapter 6, we apply the modeling approach and risk measures to insurance claims consisting of losses and loss adjustment expenses. Chapter 7 contains the concluding remarks and possible directions for further research.

[1]http://www.cbc.ca/news/canada/story/2009/04/08/f-mathwhiz.html

Chapter 2

Multivariate Distributions and Copulas

A copula function is a joint distribution function with marginal distribution functions as parameters. Therefore, properties of copulas are analogous to those of joint distributions. Comprehensive discussions of multivariate distributions and copula functions and their statistical properties can be found in monographs by Joe (1997) and Nelsen (2006). This chapter outlines the basic properties and results useful for our models in later chapters. Section 2.1 gives a brief summary of the propensities of joint distributions. Copulas are introduced in Section 2.2.

2.1 Basics of Joint Distributions

The joint distribution of n random variables $X_1, X_2, ..., X_n$ is defined as the function F whose value at every point $(x_1, x_2, ..., x_n)$ in n-dimensional space R^n is specified by

$$F(x_1, x_2, ..., x_n) = P(X_i \leq x_i;\ i = 1, 2, ..., n),$$

and the survival function corresponding to $F(x_1, x_2, ..., x_n)$ is given by

$$S(x_1, x_2, ..., x_n) = P(X_i > x_i;\ i = 1, 2, ..., n).$$

2.1.1 Bivariate distributions

Without loss of generality, this project focuses on the joint behavior of two random variables. Necessary and sufficient conditions for a right-continuous function F to be a bivariate distribution function are:

(i) $\lim_{x_i \to -\infty} F(x_1, x_2) = 0$, for any $i = 1, 2$;

(ii) $\lim_{x_i \to \infty} F(x_1, x_2) = 1$, for each $i = 1, 2$;

(iii) By the rectangle inequality, for all (a_1, a_2) and (b_1, b_2), with $a_1 \leq b_1, a_2 \leq b_2$,

$$F(b_1, b_2) - F(a_1, b_2) - F(b_1, a_2) + F(a_1, a_2) \geq 0.$$

Conditions (i) and (ii) imply that $0 \leq F \leq 1$. Condition (iii) is referred to as the property that F is 2-increasing. If F has second-order derivatives, then condition (iii) is equivalent to $\partial^2 F(x_1, x_2)/\partial x_1 \partial x_2 \geq 0$, that is, the joint density function, $f(x_1, x_2) = \partial^2 F(x_1, x_2)/\partial x_1 \partial x_2$ is non-negative.

Given the bivariate distribution function $F(x_1, x_2)$ and its density function $f(x_1, x_2)$, the marginal distribution functions F_1 and F_2 are obtained by letting $x_2 \to \infty$ and $x_1 \to \infty$, respectively. That is,

$$\begin{aligned} F_1(x_1) &= \lim_{x_2 \to \infty} F(x_1, x_2) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{x_1} f(z_1, z_2) dz_1 dz_2 \\ &= \int_{-\infty}^{x_1} f_1(z_1) dz_1, \end{aligned}$$

and

$$\begin{aligned} F_2(x_2) &= \lim_{x_1 \to \infty} F(x_1, x_2) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{x_2} f(z_1, z_2) dz_2 dz_1 \\ &= \int_{-\infty}^{x_2} f_2(z_2) dz_2, \end{aligned}$$

where $f_1(x_1)$ and $f_2(x_2)$ are marginal density functions.

For two random variables X_1 and X_2 with joint density function $f(x_1, x_2)$ and marginal densities $f_1(x_1)$ and $f_2(x_2)$, respectively, the conditional density of X_1 given $X_2 = x_2$ is given by

$$f_{1\,|\,2}(x_1\,|\,x_2) = \frac{f(x_1, x_2)}{f_2(x_2)},$$

and the conditional density of X_2 given $X_1 = x_1$ is given by

$$f_{2\,|\,1}(x_2\,|\,x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}.$$

The conditional distribution functions $F_{1\,|\,2}(x_1\,|\,x_2)$ and $F_{2\,|\,1}(x_2\,|\,x_1)$ are obtained by integrating the conditional density functions. That is,

$$\begin{aligned} F_{1\,|\,2}(x_1\,|\,x_2) &= \int_{-\infty}^{x_1} f_{1\,|\,2}(z_1\,|\,x_2)dz_1 \\ &= \int_{-\infty}^{x_1} \frac{f(z_1, x_2)}{f_2(x_2)}dz_1, \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} F_{2\,|\,1}(x_2\,|\,x_1) &= \int_{-\infty}^{x_2} f_{2\,|\,1}(z_2\,|\,x_1)dz_2 \\ &= \int_{-\infty}^{x_2} \frac{f(x_1, z_2)}{f_1(x_1)}dz_2. \end{aligned} \tag{2.2}$$

2.1.2 Fréchet-Hoeffding bounds

In this section we state the existence of maximal and minimal values of a multivariate distribution function, usually referred to as the Fréchet-Hoeffding bounds. Multivariate distribution functions take values in between these bounds on each point of their domain.

Consider multivariate distribution function $F(x_1, x_2, ..., x_n)$ with univariate marginal distribution functions $F_1, F_2, ..., F_n$. The joint distribution function is bounded below and above by the Fréchet-Hoeffding lower and upper bounds, as shown in the following theorem.

Theorem 2.1 (Fréchet-Hoeffding bounds) The Fréchet-Hoeffding lower and upper bounds F_L and F_U are defined as

$$F_L(x_1, x_2, ..., x_n) = \text{Max}\left\{\sum_{i=1}^{n} F_i(x_i) - n + 1, 0\right\},$$

$$F_U(x_1, x_2, ..., x_n) = \text{Min}\left\{F_1(x_1), F_2(x_2), ..., F_n(x_n)\right\},$$

implying

$$\text{Max}\left\{\sum_{i=1}^{n} F_i(x_i) - n + 1, 0\right\} \leq F(x_1, x_2, ..., x_n) \leq \text{Min}\{F_1(x_1), F_2(x_2), ..., F_n(x_n)\}. \tag{2.3}$$

Proof. Since

$$P\{X_1 \le x_1, X_2 \le x_2, ..., X_n \le x_n\} \le P\{X_i \le x_i\}, \quad i = 1, 2, ..., n,$$

we have

$$F(x_1, x_2, ..., x_n) \le \mathrm{Min}\{F_1(x_1), F_2(x_2), ..., F_n(x_n)\} = F_U(x_1, x_2, ..., x_n).$$

Let $A_i = \{X_i \le x_i\}$; and then $A_i^c = \{X_i > x_i\}$. Note that

$$\begin{aligned} 1 - P\{A_1 \cap A_2 \cap ... \cap A_n\} &= P\{(A_1 \cap A_2 \cap ... \cap A_n)^c\} \\ &= P\{A_1^c \cup A_2^c \cup ... \cup A_n^c\} \\ &\le P\{A_1^c\} + P\{A_2^c\} + ... + P\{A_n^c\} \\ &= 1 - P\{A_1\} + 1 - P\{A_2\} + ... + 1 - P\{A_n\}. \end{aligned}$$

As a result,

$$P\{X_1 \le x_1, X_2 \le x_2, ..., X_n \le x_n\} \ge \sum_{i=1}^{n} F_i(x_i) - n + 1,$$

which yields the left side of (2.3). □

Fréchet-Hoeffding bounds give the maximal and minimal values of multivariate distribution function. In many empirical studies, we know more about marginal distributions of related variables than their joint distribution. Fréchet-Hoeffding bounds can be used to give approximations of their joint distribution over the regions of interest.

2.2 Copulas

Since copulas are parametrically specified joint distributions generated from given marginal distributions, properties of copulas are analogous to those of joint distributions presented in the previous section. This section starts with the definition of copula and the relationship between copula and multivariate distribution, followed by the definition of the survival copula and additional properties of copulas. Families of commonly used copulas are given at the end of this section.

2.2.1 Copula functions and their connection with multivariate distributions

An n-dimensional copula function $C(u_1, u_2, ..., u_n)$ is defined as a distribution function on the unit n-cube $[0,1]^n$ which satisfies the following conditions:

(i) $C(1, ..., 1, a_k, 1, ..., 1) = a_k$ for every $1 \leq k \leq n$ and all a_k in $[0,1]$;

(ii) $C(a_1, ..., a_n) = 0$ if $a_k = 0$ for any $1 \leq k \leq n$;

(iii) C is n-increasing, that is, for all $(a_1, a_2, ..., a_n)$ and $(b_1, b_2, ..., b_n)$, with $a_i \leq b_i$, $i = 1, 2, ...n$,

$$\sum_{i_1=1}^{2} \sum_{i_2=1}^{2} ... \sum_{i_n=1}^{2} (-1)^{i_1+i_2+...+i_n} C(u_{1i_1}, u_{2i_2}, ..., u_{ni_n}) \geq 0,$$

where $u_{j1} = a_j$, $u_{j2} = b_j, j = 1, 2, ...n$.

Sklar (1959, 1973) established the unique connection between copula functions and multivariate distributions, which is known as Sklar's Theorem in copula literature.

Theorem 2.2 (Sklar) For a multivariate distribution function $F(x_1, x_2, ..., x_n)$ with univariate marginal distribution functions $F_1, F_2, ..., F_n$, there exists a unique copula C such that

$$F(x_1, x_2, ..., x_n) = C(F_1(x_1), F_2(x_2), ..., F_n(x_n)).$$

Conversely, if C is a copula, and $F_1, F_2, ..., F_n$ are univariate marginal distribution functions, then the function F defined above is a multivariate distribution function with univariate margins $F_1, F_2, ..., F_n$.

Proof. Since F_i's are univariate distribution functions, $F_i(X_i)$ follows the uniform distribution with support $[0,1]$. Let C be the joint distribution function of $F_1(X_1), F_2(X_2), ..., F_n(X_n)$. Then

$$\begin{aligned} C(u_1, u_2, ..., u_n) &= P\{F_1(X_1) \leq u_1, F_2(X_2) \leq u_2, ..., F_n(X_n) \leq u_n\} \\ &= P\{X_1 \leq F_1^{-1}(u_1), X_2 \leq F_2^{-1}(u_2), ..., X_n \leq F_n^{-1}(u_n)\} \\ &= F(F_1^{-1}(u_1), F_2^{-1}(u_2), ..., F_n^{-1}(u_n)), \end{aligned}$$

or equivalently, with $x_i = F_i^{-1}(u_i)$, $i = 1, 2, ..., n$,

$$C(F_1(x_1), F_2(x_2), ..., F_n(x_n)) = F(x_1, x_2, ..., x_n). \tag{2.4}$$

Conversely,

$$\begin{aligned} F(x_1, x_2, ..., x_n) &= P\{X_1 \leq x_1, X_2 \leq x_2, ..., X_n \leq x_n\} \\ &= P\{F_1(X_1) \leq F_1(x_1), F_2(X_2) \leq F_2(x_2), ..., F_n(X_n) \leq F_n(x_n)\} \\ &= C(F_1(x_1), F_2(x_2), ..., F_n(x_n)). \end{aligned} \tag{2.5}$$

□

The practical implication of Sklar's theorem is that copulas can be used to express a multivariate distribution in terms of its marginal distributions and the copula function. If we know a lot about the marginal distributions of individual variables, but little about their joint behavior, then copulas allow us to piece together the dependence structure of these variables.

Example 2.1 (Product copula) Let X_1 and X_2 be independent random variables. The joint distribution function is

$$F(x_1, x_2) = F_1(x_1)\, F_2(x_2).$$

Then, with $u_1 = F_1(x_1)$ and $u_2 = F_2(x_2)$,

$$\begin{aligned} C(u_1, u_2) &= C\left(F_1(x_1), F_2(x_2)\right) \\ &= F(x_1, x_2) \\ &= F_1(x_1)\, F_2(x_2) \\ &= u_1\, u_2. \end{aligned}$$

The product copula corresponds to the independence case.

To summarize, the copula approach specifies a function that binds the marginal distribution functions of random variables. The copula functions can be parameterized to include measures of dependence between the random variables. As we have seen from Example 2.1, independence is obtained by specifying a product copula. More copula functions will be introduced in subsequent sections.

2.2.2 Survival copulas

In the previous section, we have shown that any joint distribution has a unique copula representation that uses marginal distribution functions as its variables. In some empirical studies, such as statistical problems involving lifetime data or duration data, survival times are of particular interest, or the joint survival functions may be known or easier to specify. Then survival copulas might be more useful.

Suppose that function $C(u_1, u_2)$ is a copula for random variables X_1 and X_2, with $u_i = F_i(x_i)$, $i = 1, 2$. The corresponding survival copula $\widehat{C}(u_1, u_2)$ couples the joint survival function to its univariate marginal survival functions in a manner completely analogous to the way a regular copula connects the joint distribution function to its margin distribution functions.

The joint survival function of two random variables X_1 and X_2 can be related to its marginal survival functions as follows,

$$\begin{aligned} S(x_1, x_2) &= P\{X_1 > x_1, X_2 > x_2\} \\ &= 1 - F_1(x_1) - F_2(x_2) + F(x_1, x_2) \\ &= S_1(x_1) + S_2(x_2) - 1 + F(x_1, x_2) \\ &= S_1(x_1) + S_2(x_2) - 1 + C(F_1(x_1), F_2(x_2)) \\ &= S_1(x_1) + S_2(x_2) - 1 + C(1 - S_1(x_1), 1 - S_2(x_2)), \end{aligned} \tag{2.6}$$

where $S_i(x_i)$, $i = 1, 2$ are marginal survival functions, and by definition, $S_i(x_i) = 1 - F_i(x_i)$, $i = 1, 2$.

Analogous to (2.5), the copula representation of the joint survival function can be defined as

$$S(x_1, x_2) = \widehat{C}(S_1(x_1), S_2(x_2)), \tag{2.7}$$

where $\widehat{C}(u_1, u_2)$ is called the survival copula.

From (2.6) and (2.7), the regular copula and its corresponding survival copula of a bivariate distribution can be related as follows:

$$\widehat{C}(u_1, u_2) = u_1 + u_2 - 1 + C(1 - u_1, 1 - u_2). \tag{2.8}$$

We comment that the survival copula and the survival function of copula are different. The survival copula $\widehat{C}(u_1, u_2)$ as defined in (2.7) and (2.8) specifies a function that binds the marginal survival functions together, whereas the survival function of the copula is defined as $\overline{C}(u_1, u_2) = P\{U_1 > u_1, U_2 > u_2\}$. Their relationship is given by

$$\begin{aligned}\overline{C}(u_1, u_2) &= P\{U_1 > u_1, U_2 > u_2\} \\ &= 1 - u_1 - u_2 + C(u_1, u_2) \\ &= \widehat{C}(1 - u_1, 1 - u_2).\end{aligned}$$

2.2.3 Additional properties

(1) Copula density

The joint density function of copula $C(u_1, u_2)$ is given by

$$c(u_1, u_2) = \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \, \partial u_2}. \tag{2.9}$$

Now we can relate the joint density function of two random variables and its corresponding copula density as follows:

$$\begin{aligned}f(x_1, x_2) &= \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \, \partial x_2} \\ &= \frac{\partial^2 C(F_1(x_1), F_2(x_2))}{\partial x_1 \, \partial x_2} \\ &= \frac{\partial^2 C(F_1(x_1), F_2(x_2))}{\partial F_1(x_1) \, \partial F_2(x_2)} \frac{\partial F_1(x_1)}{\partial x_1} \frac{\partial F_2(x_2)}{\partial x_2} \\ &= c(u_1, u_2) f_1(x_1) f_2(x_2).\end{aligned} \tag{2.10}$$

(2) Conditioning with copula

The conditional distributions can be defined using copulas. Let $C_i(u_1, u_2)$ denote the derivative of copula function $C(u_1, u_2)$ with respect to u_i, $i = 1, 2$. The relationship between conditional distributions and partial derivatives of copula functions is detailed in the following proposition.

Proposition 2.1 (Conditioning with copula) Define

$$\begin{aligned}C_1(u_1, u_2) &= \frac{\partial C(u_1, u_2)}{\partial u_1}, \\ C_2(u_1, u_2) &= \frac{\partial C(u_1, u_2)}{\partial u_2}.\end{aligned}$$

Then

$$F_{1\,|\,2}(x_1\,|\,x_2) = C_2(u_1, u_2), \tag{2.11}$$

$$F_{2\,|\,1}(x_2\,|\,x_1) = C_1(u_1, u_2). \tag{2.12}$$

Proof. Sklar's theorem establishes that

$$C(u_1, u_2) = C(F_1(x_1), F_2(x_2)) = F(x_1, x_2).$$

By definition,

$$\begin{aligned} C_1(u_1, u_2) &= \frac{\partial C(u_1, u_2)}{\partial u_1} \\ &= \frac{\partial C(F_1(x_1), F_2(x_2))}{\partial F_1(x_1)} \\ &= \frac{\dfrac{\partial F(x_1, x_2)}{\partial x_1}}{\dfrac{\partial F_1(x_1)}{\partial x_1}}. \end{aligned}$$

Furthermore, the derivatives of the joint distribution function and the marginal distribution function are

$$\frac{\partial F(x_1, x_2)}{\partial x_1} = \frac{\partial}{\partial x_1} \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f(z_1, z_2) dz_1 dz_2 = \int_{-\infty}^{x_2} f(x_1, z_2) dz_2,$$

and

$$\frac{\partial F_1(x_1)}{\partial x_1} = f_1(x_1).$$

Thus,

$$C_1(u_1, u_2) = \frac{\int_{-\infty}^{x_2} f(x_1, z_2) dz_2}{f_1(x_1)} = \int_{-\infty}^{x_2} \frac{f(x_1, z_2)}{f_1(x_1)} dz_2 = F_{2\,|\,1}(x_2\,|\,x_1),$$

which is (2.11). (2.12) can be proven similarly. □

The implication of the above proposition is that if $C_1(u_1, u_2)$ and $C_2(u_1, u_2)$ can be inverted algebraically, then the simulation of the joint distribution can be done using the corresponding conditional distribution. That is, first simulate a value of U_1, say u_1, then simulate a value of U_2 from $C_1(u_1, u_2)$ – the conditional distribution of U_2 given $U_1 = u_1$. The detailed procedure of simulating copulas is presented in Appendix B.

(3) Fréchet-Hoeffding bounds

Because copulas are multivariate distribution functions, the Fréchet-Hoeffding bounds discussed in Section 2.1.2 also apply to copulas, that is,

$$\mathrm{Max}\left\{\sum_{i=1}^{n} u_i - n + 1, 0\right\} \leq C(u_1, u_2, ..., u_n) \leq \mathrm{Min}\{u_1, u_2, ..., u_n\}.$$

In the case of bivariate copulas, the two bounds are themselves copulas, with the lower bound

$$C_L(u_1, u_2) = \mathrm{Max}(u_1 + u_2 - 1, 0), \tag{2.13}$$

and the upper bound

$$C_U(u_1, u_2) = \mathrm{Min}(u_1, u_2). \tag{2.14}$$

The distribution of $C_L(u_1, u_2)$ has all its mass on the diagonal between $(0, 1)$ and $(1, 0)$, whereas that of $C_U(u_1, u_2)$ has its mass on the diagonal between $(0, 0)$ and $(1, 1)$. In these cases we say $C_L(u_1, u_2)$ and $C_U(u_1, u_2)$ describe perfect negative and perfect positive dependence, respectively. In probability theory, perfect positive or negative dependence is defined in terms of comonotonicity or countermonotonicity (Denuit et al., 2005).

Definition 2.1 X_1 and X_2 are comonotonic if and only if there exists a random variable Z and non-decreasing functions g_1 and g_2, such that

$$X_1 = g_1(Z), \qquad X_2 = g_2(Z).$$

Proposition 2.2 If (X_1, X_2) has copula C_U then X_1 and X_2 are said to be comonotonic. If (X_1, X_2) has copula C_L then they are said to be countermonotonic.

Proof. See Denuit et al. (2005). □

We have the following remarks: (i) X_1 and X_2 are comonotonic if and only if for any (x_1, x_2) and (x'_1, x'_2), there are either $\{x_1 \leq x'_1, x_2 \leq x'_2\}$ or $\{x_1 \geq x'_1, x_2 \geq x'_2\}$. (ii) X_1 and X_2 are countermonotonic if and only if for any (x_1, x_2) and (x'_1, x'_2), there are either $\{x_1 \leq x'_1, x_2 \geq x'_2\}$ or $\{x_1 \geq x'_1, x_2 \leq x'_2\}$.

Comonotonocity and countermonotonicy are two extreme cases of dependence. A detailed introduction of dependence measures will be given in Chapter 4.

2.3 Some Families of Copulas

A large number of copulas have been proposed in the literature, and each of these copulas imposes a different dependence structure. In this section, we discuss some copulas that have appeared frequently in empirical applications.

2.3.1 Archimedean copulas

Bivariate Archimedean copulas take the form

$$C(u_1, u_2; \gamma) = \phi^{-1}(\phi(u_1) + \phi(u_2)), \tag{2.15}$$

where ϕ is known as a generator function, and γ is the dependence parameter embedded in the function form of Archimedean generator.

A generator function that satisfies the following properties is capable of generating a valid copula,

(i) $\phi(1) = 0$;

(ii) $\phi'(s) < 0$;

(iii) $\phi''(s) > 0$.

These properties imply that $\phi(s)$ is a convex decreasing function.

(1) Product copula

Let $\phi(s) = -\ln(s)$, implying that the inverse of this generator is $\phi^{-1}(t) = e^{-t}$.

Using generator function (2.15), we obtain the product copula below:

$$\begin{aligned} C(u_1, u_2) &= e^{-(-\ln(u_1) - \ln(u_2))} \\ &= e^{\ln(u_1 u_2)} \\ &= u_1 u_2. \end{aligned}$$

(2) Clayton copula

The Clayton copula has generator function $\phi(s) = s^{-\gamma} - 1$, and the inverse of the generator is given by $\phi^{-1}(t) = (1 + t)^{-1/\gamma}$.

From the definition of Archimedean family of copulas given by (2.15), we can derive the function form of the Clayton copula:

$$\begin{aligned} C(u_1, u_2) &= (1 + u_1^{-\gamma} - 1 + u_2^{-\gamma} - 1)^{-\frac{1}{\gamma}} \\ &= (u_1^{-\gamma} + u_2^{-\gamma} - 1)^{-\frac{1}{\gamma}}. \end{aligned} \tag{2.16}$$

The dependence parameter γ takes values in the inteval $(0, \infty)$. As γ approaches zero, the copula becomes the one for the independence case. As γ approaches infinity, the copula reaches the Fréchet-Hoeffding upper bound. The Clayton copula can only account for positive dependence, and it exhibits relatively strong left tail dependence and relatively weak right tail dependence. Details of the dependence structure are shown in Chapter 4.

(3) Frank copula

The Frank copula is produced by the generator function, $\phi(s) = -\ln\left(\frac{e^{-\gamma s}-1}{e^{-\gamma}-1}\right)$. The inverse of this generator is given by $\phi^{-1}(t) = -\frac{1}{\gamma}\ln(1 + e^{-t}(e^{-\gamma} - 1))$.

By the definition of the Archimedean family of copulas in (2.15), we have the function of the Frank copula:

$$\begin{aligned} C(u_1, u_2) &= -\frac{1}{\gamma}\ln\left(1 + e^{\ln(\frac{e^{-\gamma u_1}-1}{e^{-\gamma}-1})+\ln(\frac{e^{-\gamma u_2}-1}{e^{-\gamma}-1})}(e^{-\gamma} - 1)\right) \\ &= -\frac{1}{\gamma}\ln\left(1 + \frac{(e^{-\gamma u_1} - 1)(e^{-\gamma u_2} - 1)}{e^{-\gamma} - 1}\right). \end{aligned} \tag{2.17}$$

The dependence parameter γ can take any real value in $(-\infty, \infty)$, with values $-\infty$, 0, and ∞ corresponding to the Fréchet-Hoeffding lower bound, independence, and the Fréchet-Hoeffding upper bound, respectively. The Frank copula exhibits strong dependence in the middle of the distribution, and weak tail dependence. Detailed description of the dependence structure is in Chapter 4.

2.3.2 Elliptical copulas

Elliptical copulas are associated with elliptical distributions which include the multivariate normal and multivariate t distributions.

The bivariate normal copula is given by

$$\begin{aligned} C(u_1\,, u_2) &= \Phi_\rho\left(\Phi^{-1}(u_1)\,, \Phi^{-2}(u_1)\right) \\ &= \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} e^{-\frac{s^2-2\rho st+t^2}{2(1-\rho^2)}} dsdt, \end{aligned}$$

where Φ_ρ is the distribution function of a standard bivariate normal distribution with correlation coefficient ρ, and Φ is the standard normal distribution function.

The normal copula is symmetric in both tails, and allows for both positive and negative dependence. As the dependence parameter ρ approaches -1 and 1, the bivariate normal copula attains the Fréchet-Hoeffding lower and upper bounds.

The bivariate t copula is given by

$$C(u_1\,, u_2) = \int_{-\infty}^{t_v^{-1}(u_1)} \int_{-\infty}^{t_v^{-1}(u_2)} \frac{1}{2\pi(1-\rho^2)^{1/2}} \left(1 + \frac{(s^2 - 2\rho st + t^2)}{v(1-\rho^2)}\right)^{-\frac{v+2}{2}} dsdt,$$

where t_v^{-1} is the quantile function of the univariate t distribution with v degrees of freedom, and ρ is the correlation coefficient.

The bivariate t copula is also symmetric and can capture both positive and negative dependence. Of the two parameters in the t copula, the degree of freedom v controls the heaviness of tails, while ρ measures the correlation between the two variables.

Chapter 3

Modeling Bivariate Distributions and Copulas with Common Random Effects

The modeling framework in this project is motivated by the approaches introduced in Marshall and Olkin (1988) on generating multivariate distributions by mixtures and in Oakes (1989, 1994) on frailty model. These approaches allow derivation of multivariate distributions with univariate marginal distributions as parameters, which greatly simplifies the construction of copulas. The mixture models can be used to capture a wide range of the dependence structure, as well as various levels of the tail dependence. This approach is also quite flexible and can model the joint behavior of random variables in terms of distribution functions or survival functions.

3.1 Modeling Distribution Functions

Following the presentations of the mixture model in Marshall and Olkin (1988), let X_i, $i = 1, 2$, be random variables with conditional distribution functions,

$$F_i(x_i \mid \Theta = \theta) = H_i(x_i)^{\theta},$$

where H_i is the baseline distribution function, and Θ is the common random effect that affects X_1 and X_2 simultaneously.

Assume that conditional on $\Theta = \theta$, $\{X_i \,|\, \theta,\, i = 1, 2\}$ are independent. As a result, the conditional joint distribution function can be written as

$$\begin{aligned} F(x_1, x_2 \,|\, \theta) &= F_1(x_1 \,|\, \Theta = \theta) F_2(x_2 \,|\, \Theta = \theta) \\ &= H_1(x_1)^{\theta} H_2(x_2)^{\theta}. \end{aligned}$$

The unconditional joint distribution function is

$$\begin{aligned} F(x_1, x_2) &= \int_{-\infty}^{\infty} H_1(x_1)^{\theta} H_2(x_2)^{\theta} g(\theta) d\theta \\ &= E_{\Theta} \left[H_1(x_1)^{\Theta} H_2(x_2)^{\Theta} \right], \end{aligned}$$

where $g(\theta)$ is the density function of the common random effect Θ.

Assume that the Laplace transform of the common random effect variable Θ is

$$\varphi(s) = E_{\Theta} \left[e^{-s\Theta} \right].$$

Then the unconditional joint distribution function can be written as

$$\begin{aligned} F(x_1, x_2) &= E_{\Theta} \left[H_1(x_1)^{\Theta} H_2(x_2)^{\Theta} \right] \\ &= E_{\Theta} \left[e^{\Theta \ln H_1(x_1) + \Theta \ln H_2(x_2)} \right] \\ &= \varphi \left(-\ln H_1(x_1) - \ln H_2(x_2) \right). \end{aligned} \tag{3.1}$$

That is, the joint distribution function of X_1 and X_2 can be expressed by the Laplace transform of the random effect Θ and the baseline distribution functions H_1 and H_2.

Similarly, the unconditional univariate marginal distributions can also be expressed in terms of the Laplace transform,

$$\begin{aligned} F_i(x_i) &= E_{\Theta} \left[H_i(x_i)^{\Theta} \right] \\ &= E_{\Theta} \left[e^{\Theta \ln H_i(x_i)} \right] \\ &= \varphi \left(-\ln H_i(x_i) \right), \quad i = 1, 2. \end{aligned} \tag{3.2}$$

From equation (3.2), we immediately have $-\ln H_i(x_i) = \varphi^{-1}(F_i(x_i))$, provided that the inverse of function φ exists. Then the unconditional joint distribution function given by equation (3.1) can be expressed as a function of univariate marginal distribution functions,

$$\begin{aligned} F(x_1, x_2) &= \varphi \left(-\ln H_1(x_1) - \ln H_2(x_2) \right) \\ &= \varphi \left(\varphi^{-1}(F_1(x_1)) + \varphi^{-1}(F_2(x_2)) \right). \end{aligned} \tag{3.3}$$

Following the copula representation of joint distribution functions in Chapter 2, the joint distribution function F given by equation (3.3) can then be expressed as the following copula,

$$C(u_1, u_2) = \varphi\left(\varphi^{-1}(u_1) + \varphi^{-1}(u_2)\right). \tag{3.4}$$

We remark that modeling the joint distribution function with common random effect Θ gives exactly the same results as the Archimedean approach of generating copulas introduced in (2.15). The relationship between the Archimedean generator and the Laplace transform of the common random effect is given by

$$\phi(s) = \varphi^{-1}(s). \tag{3.5}$$

Here the generator, which is the inverse of the Laplace transform of the common random effect Θ, uniquely determines an Archimedean copula.

The following subsections are devoted to various distributions of the common random effect Θ and their corresponding copulas. The dependence structure of these copulas is investigated in Chapter 4.

3.1.1 Independence

If the common random effect Θ is degenerate, the resulting joint distribution and copula correspond to independence. The Laplace transform of the degenerate distribution with constant mass at unity is

$$\varphi(s) = e^{-s}.$$

The Archimedean generator, therefore, is

$$\phi(s) = -\ln s,$$

which corresponds to the product copula, as by (3.4) we can obtain

$$\begin{aligned} C(u_1, u_2) &= e^{-(-\ln u_1 - \ln u_2)} \\ &= u_1\, u_2. \end{aligned}$$

3.1.2 Clayton copula

If the common effect Θ follows a gamma distribution with a scale parameter of 1 and a shape parameter of $1/\gamma$ (or a rate parameter of γ), the Laplace transform is given by

$$\varphi(s) = (1+s)^{-\frac{1}{\gamma}}.$$

The inverse of the Laplace transform is

$$\phi(s) = s^{-\gamma} - 1,$$

which corresponds to the Clayton copula, as by (3.4) we have

$$\begin{aligned} C(u_1, u_2) &= (1 + u_1^{-\gamma} - 1 + u_2^{-\gamma} - 1)^{-\frac{1}{\gamma}} \\ &= (u_1^{-\gamma} + u_2^{-\gamma} - 1)^{-\frac{1}{\gamma}}, \end{aligned}$$

that is (2.16) in last chapter.

3.1.3 Frank copula

The Frank copula in (2.17) can be derived from our model by letting the common random effect Θ follow a logarithmic distribution

$$f(\theta) = \frac{1}{\gamma}\frac{(1-e^{\gamma})^{\theta}}{\theta}.$$

The Laplace transform of the above logarithmic distribution is given by

$$\varphi(s) = -\frac{1}{\gamma}\ln\left(1 + e^{-s}(e^{-\gamma}-1)\right).$$

The Archimedean generator, therefore, is

$$\phi(s) = -\ln\left(\frac{e^{-\gamma s}-1}{e^{-\gamma}-1}\right),$$

which corresponds to the Frank copula, since by (3.4) we obtain

$$\begin{aligned} C(u_1, u_2) &= -\frac{1}{\gamma}\ln\left(1 + e^{\ln\left(\frac{e^{-\gamma u_1}-1}{e^{-\gamma}-1}\right) + \ln\left(\frac{e^{-\gamma u_2}-1}{e^{-\gamma}-1}\right)}(e^{-\gamma}-1)\right) \\ &= -\frac{1}{\gamma}\ln\left(1 + \frac{(e^{-\gamma u_1}-1)(e^{-\gamma u_2}-1)}{e^{-\gamma}-1}\right). \end{aligned}$$

3.1.4 Gumbel copula

The Gumbel copula was originally studied by Gumbel (1960), and can be found in empirical work like Hougaard (1986a). Details of its dependence structure is explored in Chapter 4. The focus of this section is the derivation of the Gumbel copula using common random effects. If the common random effect Θ follows a positive stable distribution with probability density function

$$f(\theta) = -\frac{1}{\pi\theta}\sum_{k=1}^{\infty}\frac{\Gamma(1+\frac{k}{\gamma})}{k!}\left(-\theta^{-\frac{1}{\gamma}}\right)^k \sin\left(\frac{k\pi}{\gamma}\right),$$

then the resulting multivariate distribution has the representation of the Gumbel copula.

The Laplace transform of the above positive stable distribution is given by

$$\varphi(s) = e^{-s^{\frac{1}{\gamma}}}.$$

The Archimedean generator, therefore, is

$$\phi(s) = (-\ln s)^{\gamma},$$

which corresponds to the Gumbel copula, given by

$$C(u_1, u_2) = e^{-\left((-\ln u_1)^{\gamma}+(-\ln u_2)^{\gamma}\right)^{\frac{1}{\gamma}}}. \tag{3.6}$$

Table 3.1 summarizes the above mentioned popular copulas and their generators.

Table 3.1: Archimedean Copulas and Their Generators

	Generator $\phi(s)$	**Laplace transform of Θ**	**Range of γ**	**Bivariate copula $C(u_1, u_2)$**
Independence	$-\ln(s)$	e^{-s}	Not applicable	$u_1 u_2$
Clayton	$s^{-\gamma}-1$	$(1+s)^{-1/\gamma}$	$\gamma > 0$	$(u_1^{-\gamma}+u_2^{-\gamma}-1)^{-1/\gamma}$
Frank	$-\ln\left(\frac{e^{-\gamma s}-1}{e^{-\gamma}-1}\right)$	$-\frac{1}{\gamma}\ln(1+e^{-s}(e^{-\gamma}-1))$	$-\infty < \gamma < \infty$	$-\frac{1}{\gamma}\ln\left(1+\frac{(e^{-\gamma u_1}-1)(e^{-\gamma u_2}-1)}{e^{-\gamma}-1}\right)$
Gumbel	$(-\ln s)^{\gamma}$	$e^{-s^{1/\gamma}}$	$\gamma \geq 1$	$e^{-\left((-\ln u_1)^{\gamma}+(-\ln u_2)^{\gamma}\right)^{1/\gamma}}$

3.2 Modeling Survival Functions

In empirical studies that involve duration data and lifetime data, working with survival functions would be both natural and convenient. We use the ideas proposed in Marshall and Olkin (1988) for constructing multivariate distributions with mixtures and apply the same methodology to the joint survival functions and their univariate marginals.

Let X_i, $i = 1, 2$, be random variables with conditional survival function

$$S_i(x_i \,|\, \Theta = \theta) = B_i(x_i)^{\theta},$$

where B_i is the baseline survival function, and Θ is the common random effect.

Assume that conditional on $\Theta = \theta$, $\{X_i \,|\, \theta,\ i = 1, 2\}$ are independent. As a result, the conditional joint survival function can be written as

$$\begin{aligned} S(x_1, x_2 \,|\, \theta) &= S_1(x_1 \,|\, \Theta = \theta) S_2(x_2 \,|\, \Theta = \theta) \\ &= B_1(x_1)^{\theta} B_2(x_2)^{\theta}. \end{aligned}$$

The unconditional joint survival function is

$$\begin{aligned} S(x_1, x_2) &= \int_{-\infty}^{\infty} B_1(x_1)^{\theta} B_2(x_2)^{\theta} g(\theta) d\theta \\ &= E_{\Theta} \left[B_1(x_1)^{\Theta} B_2(x_2)^{\Theta} \right], \end{aligned}$$

where $g(\theta)$ is the density function of the common random effect Θ.

Assume that the Laplace transform of the common random effect variable Θ is

$$\varphi(s) = E_{\Theta} \left[e^{-s\Theta} \right].$$

Then, the unconditional joint survival function can be written as

$$\begin{aligned} S(x_1, x_2) &= E_{\Theta} \left[B_1(x_1)^{\Theta} B_2(x_2)^{\Theta} \right] \\ &= E_{\Theta} \left[e^{\Theta \ln B_1(x_1) + \Theta \ln B_2(x_2)} \right] \\ &= \varphi \left(-\ln B_1(x_1) - \ln B_2(x_2) \right). \end{aligned} \tag{3.7}$$

That is, the joint survival function of X_1 and X_2 can be expressed by the Laplace transform of the common random effect Θ and the baseline survival functions B_1 and B_2.

Similarly, the unconditional univariate marginal survival functions can also be expressed in terms of the Laplace transform,

$$
\begin{aligned}
S_i(x_i) &= E_\Theta\left[B_i(x_i)^\Theta\right] \\
&= E_\Theta\left[e^{\Theta \ln B_i(x_i)}\right] \\
&= \varphi\left(-\ln B_i(x_i)\right), \quad i = 1, 2.
\end{aligned} \tag{3.8}
$$

From equation (3.8), we have $-\ln B_i(x_i) = \varphi^{-1}(S_i(x_i))$, given that the inverse of function φ exists. Then the unconditional joint survival function given by equation (3.7) can be expressed as a function of univariate marginal survival functions,

$$
\begin{aligned}
S(x_1, x_2) &= \varphi\left(-\ln B_1(x_1) - \ln B_2(x_2)\right) \\
&= \varphi\left(\varphi^{-1}(S_1(x_1)) + \varphi^{-1}(S_2(x_2))\right).
\end{aligned} \tag{3.9}
$$

Then the joint distribution function can be written as

$$
\begin{aligned}
F(x_1, x_2) &= F_1(x_1) + F_2(x_2) - 1 + S(x_1, x_2) \\
&= F_1(x_1) + F_2(x_2) - 1 + \varphi(\varphi^{-1}(1 - F_1(x_1)) + \varphi^{-1}(1 - F_2(x_2))).
\end{aligned} \tag{3.10}
$$

From (3.9), we have the survival copula representation

$$
\widehat{C}(u_1, u_2) = \varphi\left(\varphi^{-1}(u_1) + \varphi^{-1}(u_2)\right), \tag{3.11}
$$

which corresponds to the survival copula introduced in (2.7).

Furthermore, the joint distribution function F given by equation (3.10) can also be expressed as the following regular copula,

$$
C(u_1, u_2) = u_1 + u_2 - 1 + \varphi(\varphi^{-1}(1 - u_1) + \varphi^{-1}(1 - u_2)). \tag{3.12}
$$

The copulas given by (3.11) and (3.12) are related through equation (2.8), which means that the two approaches of modeling distribution functions and modeling survival functions are symmetric. Given the marginal survival functions and the regular copula, the joint survival distribution and survival copula can be obtained. Given the marginal distribution functions and the survival copula, the joint distribution and regular copula can be obtained.

In the following subsections, we present the various distributions of the common effect Θ and the corresponding copulas based on the modeling of survival functions.

3.2.1 Independence

Same as the results from modeling the distribution functions, if the common random effect Θ is degenerate, the resulting multivariate survival function and copula correspond to independence. The Laplace transform of the degenerate distribution with constant mass at unity is

$$\varphi(s) = e^{-s}.$$

The Archimedean generator, therefore, is

$$\phi(s) = -\ln s,$$

which corresponds to the product copula by (3.12) that

$$\begin{aligned} C(u_1, u_2) &= u_1 + u_2 - 1 + e^{-(-\ln(1-u_1)-\ln(1-u_2))} \\ &= u_1 + u_2 - 1 + (1-u_1)(1-u_2) \\ &= u_1\, u_2. \end{aligned}$$

3.2.2 Pareto copula

Now let the common random effect Θ follows a gamma distribution with a scale parameter of 1 and a shape parameter of $1/\gamma$ (or a rate parameter of γ). The Laplace transform of the gamma distribution is given by

$$\varphi(s) = (1+s)^{-\frac{1}{\gamma}}.$$

The Archimedean generator, therefore, is

$$\phi(s) = s^{-\gamma} - 1,$$

which corresponds to the following Pareto copula, as by (3.12) we can obtain

$$\begin{aligned} C(u_1, u_2) &= u_1 + u_2 - 1 + (1 + (1-u_1)^{-\gamma} - 1 + (1-u_2)^{-\gamma} - 1)^{-\frac{1}{\gamma}} \\ &= u_1 + u_2 - 1 + \big((1-u_1)^{-\gamma} + (1-u_2)^{-\gamma} - 1\big)^{-\frac{1}{\gamma}}. \end{aligned} \tag{3.13}$$

3.2.3 Frank copula

The Frank copula is invariant to the choice of distribution function or survival function. Assuming that the common random effect Θ follows a logarithmic distribution as given in Section 3.1.3, then the Laplace transform has the form

$$\varphi(s) = -\frac{1}{\gamma}\ln(1 + e^{-s}(e^{-\gamma} - 1)).$$

The Archimedean generator, therefore, is

$$\phi(s) = -\ln\left(\frac{e^{-\gamma s} - 1}{e^{-\gamma} - 1}\right),$$

which corresponds to the Frank copula, as by (3.12) we have

$$\begin{aligned}
C(u_1, u_2) &= u_1 + u_2 - 1 - \frac{1}{\gamma}\ln\left(1 + e^{\ln\left(\frac{e^{-\gamma(1-u_1)}-1}{e^{-\gamma}-1}\right)+\ln\left(\frac{e^{-\gamma(1-u_2)}-1}{e^{-\gamma}-1}\right)}(e^{-\gamma} - 1)\right) \\
&= u_1 + u_2 - 1 - \frac{1}{\gamma}\ln\left(1 + \frac{\left(e^{-\gamma(1-u_1)} - 1\right)\left(e^{-\gamma(1-u_2)} - 1\right)}{e^{-\gamma} - 1}\right) \\
&= -\frac{1}{\gamma}\ln\left(e^{-\gamma(u_1+u_2-1)}\right) - \frac{1}{\gamma}\ln\left(1 + \frac{\left(e^{-\gamma(1-u_1)} - 1\right)\left(e^{-\gamma(1-u_2)} - 1\right)}{e^{-\gamma} - 1}\right) \\
&= -\frac{1}{\gamma}\ln\left(e^{-\gamma(u_1+u_2-1)} + \frac{e^{-\gamma(u_1+u_2-1)}\left(e^{-\gamma(1-u_1)} - 1\right)\left(e^{-\gamma(1-u_2)} - 1\right)}{e^{-\gamma} - 1}\right) \\
&= -\frac{1}{\gamma}\ln\left(\frac{(e^{-\gamma} - 1) + (e^{-\gamma u_1} - 1)(e^{-\gamma u_2} - 1)}{e^{-\gamma} - 1}\right) \\
&= -\frac{1}{\gamma}\ln\left(1 + \frac{(e^{-\gamma u_1} - 1)(e^{-\gamma u_2} - 1)}{e^{-\gamma} - 1}\right).
\end{aligned}$$

3.2.4 Hougaard copula

The Hougaard copula, which was proposed in Hougaard (1986b, 1987), can be constructed through modeling survival functions and assuming that the common random effect follows a positive stable distribution given in Section 3.1.4.

The Laplace transform of this positive stable distribution is given by

$$\varphi(s) = e^{-s^{1/\gamma}}.$$

The Archimedean generator, therefore, is

$$\phi(s) = (-\ln s)^{\gamma},$$

which corresponds to the Hougaard copula, given by

$$C(u_1, u_2) = u_1 + u_2 - 1 + e^{-[(-\ln(1-u_1))^\gamma + (-\ln(1-u_2))^\gamma]^{1/\gamma}}.$$

Generally speaking, modeling distribution functions and modeling survival functions yield different joint distributions because

$$P(X \le x \,|\, \Theta = \theta) = H(x)^\theta \ne (1 - B(x))^\theta.$$

The Frank copula is symmetric around $(1/2\,,1/2)$. Thus it is invariant to the choice of distribution function or survival function in the common random effect model. The product copula is also invariant to the choice of functions. Gamma and positive stable families of the common random effect yield different bivariate distributions and therefore different copulas. Table 3.2 summarizes the differences between the two approaches of modeling the joint behavior of random variables. Properties of these copulas and their dependence structure are discussed in Chapter 4.

Table 3.2: Two Approaches of Modeling Joint Behavior of Random Variables

Laplace transform of the common effect Θ	**Distribution of Θ**	**Modeling distribution function**	
		Copula type	Copula function $C(u_1, u_2)$
e^{-s}	Degenerate	Product (independence)	$u_1 u_2$
$(1+s)^{-1/\gamma}$	Gamma	Clayton	$(u_1^{-\gamma} + u_2^{-\gamma} - 1)^{-1/\gamma}$
$-\frac{1}{\gamma}\ln(1 + e^{-s}(e^{-\gamma} - 1))$	Logarithmic	Frank	$-\frac{1}{\gamma}\ln\left(1 + \frac{(e^{-\gamma u_1}-1)(e^{-\gamma u_2}-1)}{e^{-\gamma}-1}\right)$
$e^{-s^{1/\gamma}}$	Positive stable	Gumbel	$e^{-\left((-\ln u_1)^\gamma + (-\ln u_2)^\gamma\right)^{1/\gamma}}$
Laplace transform of the common effect Θ	**Distribution of Θ**	**Modeling survival function**	
		Copula type	Copula function $C(u_1, u_2)$
e^{-s}	Degenerate	Product (independence)	$u_1 u_2$
$(1+s)^{-1/\gamma}$	Gamma	Pareto	$u_1 + u_2 - 1 + \left((1-u_1)^{-\gamma} + (1-u_2)^{-\gamma} - 1\right)^{-1/\gamma}$
$-\frac{1}{\gamma}\ln(1 + e^{-s}(e^{-\gamma} - 1))$	Logarithmic	Frank	$-\frac{1}{\gamma}\ln\left(1 + \frac{(e^{-\gamma u_1}-1)(e^{-\gamma u_2}-1)}{e^{-\gamma}-1}\right)$
$e^{-s^{1/\gamma}}$	Positive stable	Hougaard	$u_1 + u_2 - 1 + e^{-\left((-\ln(1-u_1))^\gamma + (-\ln(1-u_2))^\gamma\right)^{1/\gamma}}$

Chapter 4

Measuring Dependence

Given the wide selection of copula models, how should one model be chosen over the others in empirical work? One of the key considerations is the nature of the dependence captured by different copulas. The nature of the dependence captured by the dependence parameter(s) varies from one copula to another. Moreover, commonly used measures of dependence are related to the parameter(s) in copula functions. This chapter starts with a brief review of the widely used measures of dependence – linear correlation and rank correlation, followed by the application of measures of rank correlation to the models with common random effects. Measures of tail dependence will also be discussed.

4.1 Review of Dependence Measures

This section reviews the commonly used measures of dependence in statistics literature. We focus on the dependence measures that have appeared more often in empirical work, instead of a complete list of all the measures of dependence.

Two random variables X_1 and X_2 are said to dependent or associated if they are not independent in the sense that $F(x_1, x_2) = F_1(x_1)F_2(x_2)$, or $S(x_1, x_2) = S_1(x_1)S_2(x_2)$. Let $\delta(X_1, X_2)$ denote a scalar measure of dependence. Embrechts et al. (2002) listed four desirable properties of dependence measure:

(i) Symmetry: $\delta(X_1, X_2) = \delta(X_2, X_1)$;

(ii) Normalization: $-1 \leq \delta(X_1, X_2) \leq +1$;

(iii) $\delta(X_1, X_2) = +1$ if and only if (X_1, X_2) are comonotonic; $\delta(X_1, X_2) = -1$ if and only if (X_1, X_2) are countermonotonic;

(iv) For a strictly monotonic transformation $T : \mathbb{R} \longrightarrow \mathbb{R}$ of X_1:

$$\delta(T(X_1), X_2) = \begin{cases} \delta(X_1, X_2) & \text{if T is increasing,} \\ -\delta(X_1, X_2) & \text{if T is decreasing.} \end{cases}$$

4.1.1 Correlation coefficient

The most commonly used measure of dependence (or association) between two random variables X_1 and X_2 is Pearson's correlation coefficient, which is defined as

$$\rho_{X_1X_2} = \frac{\text{Cov}\,(X_1, X_2)}{[\text{Var}(X_1)]^{\frac{1}{2}}\,[\text{Var}(X_2)]^{\frac{1}{2}}}, \tag{4.1}$$

where $\text{Cov}(X_1, X_2) = E[X_1X_2] - E[X_1]E[X_2]$, $\text{Var}(X_1)$ and $\text{Var}(X_2)$ are the variances of X_1 and X_2, respectively.

It is well known that:

(i) $\rho_{X_1X_2}$ is a measure of linear dependence,

(ii) $\rho_{X_1X_2}$ is symmetric,

(iii) the lower and upper bounds on the inequality $-1 \leq \rho_{X_1X_2} \leq +1$ measure perfect negative and positive linear dependence, and

(iv) it is invariant with respect to strictly increasing linear transformations of the variables.

The weakness of using the correlation coefficient as a measure of dependence includes:

(i) in general, zero correlation does not imply independence,

(ii) it is not defined for heavy-tail distributions whose second moments do not exist,

(iii) it is not invariant under strictly increasing nonlinear transformations, and

(iv) attainable values of the correlation coefficients within interval $[-1, +1]$ between two variables depend upon their respective marginal distributions.

These limitations have motivated alternative measures of dependence based on ranks.

4.1.2 Rank correlation

Consider two random variables X_1 and X_2 with continuous distribution functions F_1 and F_2, respectively, and joint distribution function F. Two well-established measures of rank correlation are Spearman's rho and Kendall's tau.

Spearman's rho is the linear correlation between the distribution functions, defined as

$$\rho_S(X_1, X_2) = \rho(F_1(X_1), F_2(X_2)),$$

where $\rho = \rho_{X_1 X_2}$ is defined in (4.1).

Kendall's tau is defined as

$$\tau_K(X_1, X_2) = P\{(X_1 - X_1')(X_2 - X_2') > 0\} - P\{(X_1 - X_1')(X_2 - X_2') < 0\}, \tag{4.2}$$

where (X_1, X_2) and (X_1', X_2') are two independent pairs of random variables from F. The first term on the right hand side of equation (4.2) is referred to as the probability of concordance, and the second term as the probability of discordance, and hence

$$\tau_K(X_1, X_2) = P\{\text{concordance}\} - P\{\text{discordance}\}.$$

The similarity between Spearman's rho and Kendall's tau is that both of them measure monotonic dependence between random variables, and both are based on the concept of concordance, which refers to the property that large values of one random variable are associated with large values of another, whereas discordance refers to large values of one being associated with small values of the other.

These two well-established measures of rank correlation have properties of symmetry, normalization, comonotonicity and countermonotonicity, and both assume the value of zero under independence. Further,

$$\begin{aligned}
&\rho_S(X_1, X_2) = \tau_K(X_1, X_2) = -1 \quad \text{if and only if} \quad C = C_L = \text{Max}(u_1 + u_2 - 1, 0),\\
&\rho_S(X_1, X_2) = \tau_K(X_1, X_2) = +1 \quad \text{if and only if} \quad C = C_U = \text{Min}(u_1, u_2),\\
&\rho_S(X_1, X_2) = \tau_K(X_1, X_2) = 0 \qquad \text{if and only if} \quad C = u_1 u_2.
\end{aligned}$$

4.2 Measures of Rank Correlation for Models of Common Random Effects

Spearman's rho and Kendall's tau can be expressed in terms of copulas as follows:

$$\rho_S(X_1, X_2) = \rho_S(C) = 12 \int_0^1 \int_0^1 C(u_1, u_2) du_1 du_2 - 3; \tag{4.3}$$

$$\tau_K(X_1, X_2) = \tau_K(C) = 4\int_0^1\int_0^1 C(u_1, u_2)\frac{\partial^2 C(u_1, u_2)}{\partial u_1\,\partial u_2}du_1du_2 \,-\, 1 \tag{4.4a}$$

$$= 1 - 4\int_0^1\int_0^1 \frac{\partial C(u_1, u_2)}{\partial u_1}\frac{\partial C(u_1, u_2)}{\partial u_2}du_1du_2. \tag{4.4b}$$

More details about (4.3) and (4.4) can be found in Joe (1997) or Nelsen (2006).

For the two approaches of modeling the joint behavior of random variables in Chapter 3, the following proposition shows that the regular copula and its associated survival copula have same rank correlations.

Proposition 4.1 The rank correlation of the survival copula $\widehat{C}$ is equal to that of the regular copula C, that is,

$$\rho_S(C) = \rho_S(\widehat{C}), \tag{4.5}$$

$$\tau_K(C) = \tau_K(\widehat{C}), \tag{4.6}$$

where the survival copula $\widehat{C}$ is defined by equation (2.8).

Proof. From (2.8) and the expression of Spearman's rho in terms of copulas in (4.3), we have

$$\begin{aligned}
\rho_S(\widehat{C}) &= 12\int_0^1\int_0^1 \widehat{C}(u_1, u_2)du_1du_2 - 3 \\
&= 12\int_0^1\int_0^1 (u_1 + u_2 - 1 + C(1-u_1, 1-u_2))du_1du_2 - 3 \\
&= 12\int_0^1\int_0^1 (1 - u_1 - u_2 + C(u_1, u_2))du_1du_2 - 3 \\
&= 12\int_0^1\int_0^1 (1 - u_1 - u_2)du_1du_2 + 12\int_0^1\int_0^1 C(u_1, u_2)du_1du_2 - 3 \\
&= 12\int_0^1\int_0^1 C(u_1, u_2)du_1du_2 - 3 \\
&= \rho_S(C),
\end{aligned}$$

because in the third last line, $\int_0^1\int_0^1(1 - u_1 - u_2)du_1du_2 = 0$.

Similarly, using (2.8), (4.4b), and the properties of copulas in Section 2.2.1, we obtain

$$\begin{aligned}
\tau_K(\widehat{C}) &= 1 - 4\int_0^1\int_0^1 \frac{\partial \widehat{C}(u_1,u_2)}{\partial u_1}\frac{\partial \widehat{C}(u_1,u_2)}{\partial u_2} du_1 du_2 \\
&= 1 - 4\int_0^1\int_0^1 \left(\frac{\partial C(1-u_1,1-u_2)}{\partial(1-u_1)} - 1\right)\left(\frac{\partial C(1-u_1,1-u_2)}{\partial(1-u_2)} - 1\right) du_1 du_2 \\
&= 1 - 4\int_0^1\int_0^1 \left(\frac{\partial C(u_1,u_2)}{\partial u_1} - 1\right)\left(\frac{\partial C(u_1,u_2)}{\partial u_2} - 1\right) du_1 du_2 \\
&= 1 - 4\int_0^1\int_0^1 \frac{\partial C(u_1,u_2)}{\partial u_1}\frac{\partial C(u_1,u_2)}{\partial u_2} du_1 du_2 \\
&\quad + 4\int_0^1\int_0^1 \frac{\partial C(u_1,u_2)}{\partial u_1} du_1 du_2 + 4\int_0^1\int_0^1 \frac{\partial C(u_1,u_2)}{\partial u_2} du_1 du_2 - 4\int_0^1\int_0^1 du_1 du_2 \\
&= 1 - 4\int_0^1\int_0^1 \frac{\partial C(u_1,u_2)}{\partial u_1}\frac{\partial C(u_1,u_2)}{\partial u_2} du_1 du_2 \\
&= \tau_K(C).
\end{aligned}$$

□

Unlike Pearson's correlation coefficient, rank correlations depend on the copula of a bivariate distribution and not on the functional forms of the marginal distributions. In other words, each copula specifies a unique dependence structure and the rank correlation is a function of the dependence parameter(s) embedded in the copula. Because of the limited dependence parameter space, the Clayton, Pareto, Gumbel, and Hougaard copulas permit only non-negative association, while the Frank copula allows positive as well as negative association.

Furthermore, Kendall's tau can be evaluated directly from the Laplace transform of the common random effect Θ, as shown in the following theorem.

Theorem 4.1 Let X_1 and X_2 be random variables with copulas generated by the models of common random effects (3.4) or (3.11). Then Kendall's tau is given by

$$\tau_K(X_1,X_2) = 1 + 4\int_0^1 \varphi'(\varphi^{-1}(s))\,\varphi^{-1}(s) ds,$$

where $\varphi(s)$ is the Laplace transform of the common random effect.

Proof. Genest and MacKay (1986) gave the following expression for Kendall's tau,

$$\tau_K(X_1,X_2) = 1 + 4\int_0^1 \frac{\phi(s)}{\phi'(s)} ds,$$

where $\phi(s)$ is the Archimedean generator.

Since the Archimedean generator, $\phi(s)$, is the inverse of the Laplace transform of the common random effect, $\varphi(s)$, that is, $\phi(s) = \varphi^{-1}(s)$, then using the formula for the derivative of an inverse function, we have

$$\frac{\phi(s)}{\phi'(s)} = \varphi'(\varphi^{-1}(s))\,\varphi^{-1}(s).$$

The desired result follows immediately. □

Table 4.1 illustrates Spearman's rho and Kendall's tau for the copulas specified in Chapter 3.

Table 4.1: Copulas and Their Rank Correlations

Copula type	**Copula Function** $C(u_1, u_2)$	**Spearman's rho**	**Kendall's tau**
Product	$u_1 u_2$	0	0
Clayton	$(u_1^{-\gamma} + u_2^{-\gamma} - 1)^{-1/\gamma}$	Complicated form	$\frac{\gamma}{2+\gamma}$
Frank	$-\frac{1}{\gamma}\ln\left(1 + \frac{(e^{-\gamma u_1}-1)(e^{-\gamma u_2}-1)}{e^{-\gamma}-1}\right)$	$1 + \frac{12}{\gamma}\{D_2(\gamma) - D_1(\gamma)\}$	$1 + \frac{4}{\gamma}\{D_1(\gamma) - 1\}$
Gumbel	$e^{-\left((-\ln u_1)^\gamma + (-\ln u_2)^\gamma\right)^{1/\gamma}}$	No closed form	$1 - \gamma^{-1}$
Hougaard	$u_1 + u_2 - 1 + e^{-\left((-\ln(1-u_1))^\gamma + (-\ln(1-u_2))^\gamma\right)^{1/\gamma}}$	No closed form	$1 - \gamma^{-1}$
Pareto	$u_1 + u_2 - 1 + \left((1-u_1)^{-\gamma} + (1-u_2)^{-\gamma} - 1\right)^{-1/\gamma}$	Complicated form	$\frac{\gamma}{2+\gamma}$

The dependence measures of Frank copula depend on Debye faction, defined as $D_k(x) = \frac{k}{x^k}\int_0^x \frac{t^k}{e^t-1}dt$, for $k = 1, 2$. $D_k(-x) = D_k(x) + \frac{kx}{k+1}$

Figure 4.1 shows the scatter plots for bivariate distributions with identical marginal exponential distributions (with mean of 1) and identical rank correlation but different dependence structures. Perspective plots of the corresponding copula densities are given in Figure 4.2. If these random variables represent the insurance losses, then the Gumbel and Pareto copulas would be preferable models for insurers since extreme losses have tendency to occur together. Measures of tail dependency discussed in next section can be used to capture the extremal dependence.

4.3 Measures of Tail Dependency

As we have seen from Figures 4.1 and 4.2, copulas with same rank correlation may have dramatically different tail behavior. Measures of tail dependence may help to distinguish

Figure 4.1: Simulated Samples from Five Copulas with Same Marginal Distributions (Exponential with Mean of 1) and Same Rank Rorrelation (Kendall's tau = 0.50)

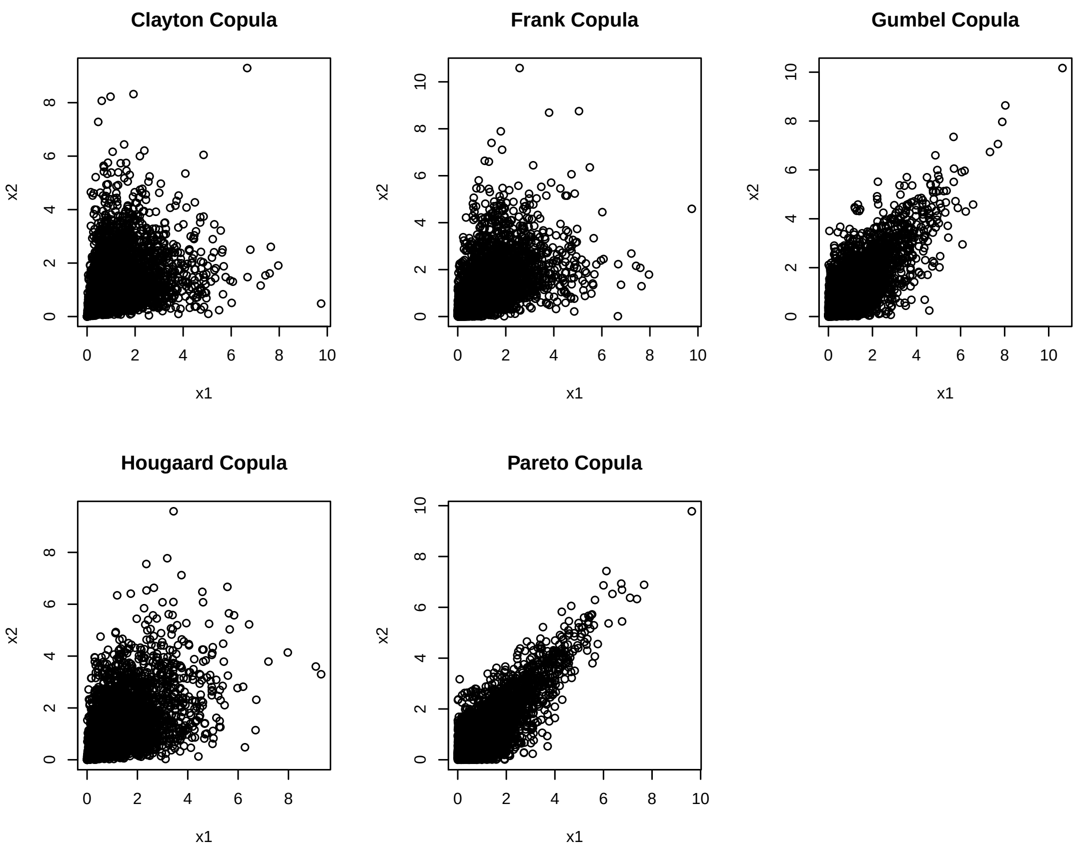
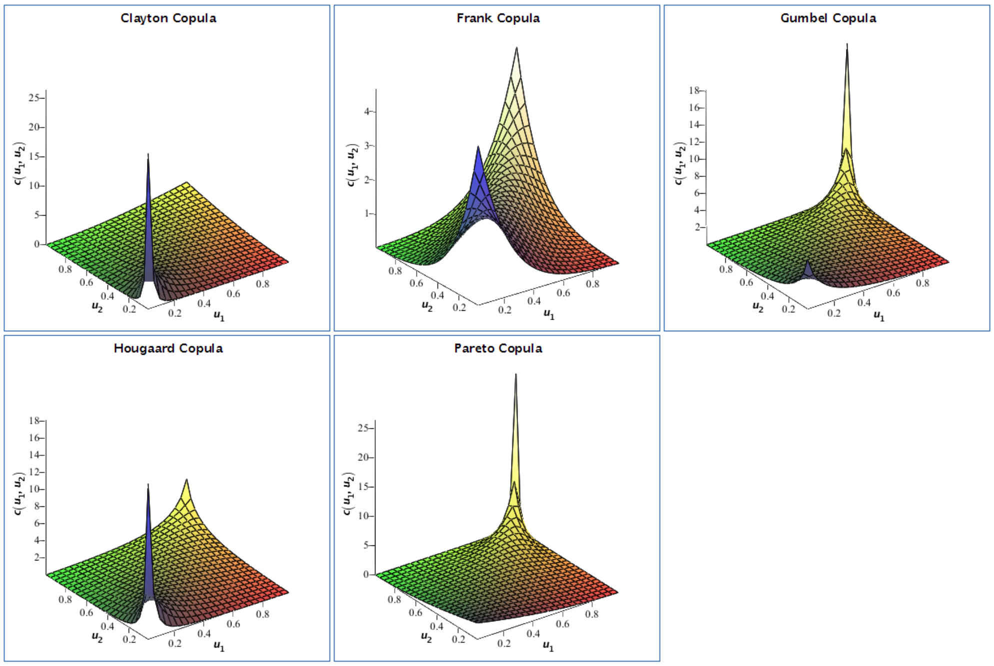


Figure 4.2: Five Copulas with the Same Rank Correlation (Kendall's tau = 0.50)



copulas. In some empirical applications, the joint behavior of tail values of random variables is of particular interest. For example, investors may be more concerned about the probability that the rates of returns of all securities in a portfolio fall below given levels. This requires measures of tail dependency. The tail dependency measure can be defined in terms of conditional probability that one random variable exceeds some value given that another exceeds some value. Various measures of tail dependency can be found in Joe (1997), Nelsen (2006), Venter (2002), and Frahm (2006).

4.3.1 Tail concentration functions

Let X_1 and X_2 be random variables with continuous distribution functions F_1 and F_2, and copula C. Then $U_1 = F_1(X_1)$ and $U_2 = F_2(X_2)$ are standard uniform random variables. The right and left tail concentration functions can be defined with reference to how much probability is in regions near $(1,1)$ and $(0,0)$. For any z in $(0,1)$, the left tail concentration function is defined as

$$\begin{aligned} L(z) &= P(U_1 < z \,|\, U_2 < z) \\ &= \frac{P(U_1 < z, U_2 < z)}{P(U_2 < z)} \\ &= \frac{C(z,z)}{z}, \end{aligned} \tag{4.7}$$

and the right tail concentration function is

$$\begin{aligned} R(z) &= P(U_1 > z \,|\, U_2 > z) \\ &= \frac{P(U_1 > z, U_2 > z)}{P(U_2 > z)} \\ &= \frac{1 - P(U_1 < z) - P(U_2 < z) + P(U_1 < z, U_2 < z)}{1 - P(U_2 < z)} \\ &= \frac{1 - 2z + C(z,z)}{1 - z}. \end{aligned} \tag{4.8}$$

The relationship between the tail concentration functions of regular copula and its associated survival copula is detailed in the following proposition.

Proposition 4.2 Let $\widehat{C}$ be the survival copula associated with the regular copula C. Then the left (right) tail concentration function of $\widehat{C}$ is equal to the right (left) tail concentration function of C, that is

$$L_{\widehat{C}}(z) = R_C(1-z), \qquad R_{\widehat{C}}(z) = L_C(1-z).$$

Proof. By the definition of the left tail concentration function in (4.7),

$$\begin{aligned} L_{\widehat{C}}(z) &= \frac{\widehat{C}(z,z)}{z} \\ &= \frac{2z - 1 + C(1-z, 1-z)}{1 - (1-z)} \\ &= R_C(1-z); \end{aligned}$$

Similarly,

$$\begin{aligned} R_{\widehat{C}}(z) &= \frac{1 - 2z + \widehat{C}(z,z)}{1-z} \\ &= \frac{C(1-z, 1-z)}{1-z} \\ &= L_C(1-z). \end{aligned}$$

□

4.3.2 Upper (lower) tail dependence coefficients

The degree of extreme co-movements of random variables can be defined by taking limits of equations (4.7) and (4.8). The upper (lower) tail dependence coefficients capture the probability that one event is extermal conditional on another extreme event, which are given by

$$\begin{aligned}\lambda_L &= \lim_{z\to 0} P(U_1 < z | U_2 < z) \\ &= \lim_{z\to 0} \frac{C(z,z)}{z},\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}\lambda_R &= \lim_{z\to 1} P(U_1 > z | U_2 > z) \\ &= \lim_{z\to 1} \frac{1-2z+C(z,z)}{1-z}.\end{aligned} \tag{4.10}$$

If λ_R (λ_L) is positive, then the two variables are said to be right (left) tail dependent, with larger values indicating stronger dependence.

4.3.3 Extremal dependence coefficients

Let $U_{\text{Min}} = \text{Min}\{U_1, U_2\}$, and $U_{\text{Max}} = \text{Max}\{U_1, U_2\}$. The lower extremal dependence coefficient is defined as

$$\begin{aligned}\epsilon_L &= \lim_{z\to 0} P(U_{\text{Max}} < z \,|\, U_{\text{Min}} < z) \\ &= \lim_{z\to 0} \frac{P(U_{\text{Max}} < z, U_{\text{Min}} < z)}{P(U_{\text{Min}} < z)} \\ &= \lim_{z\to 0} \frac{P(U_1 < z, U_2 < z)}{1 - P(U_1 > z, U_2 > z)} \\ &= \lim_{z\to 0} \frac{P(U_1 < z, U_2 < z)}{P(U_1 < z) + P(U_2 < z) - P(U_1 < z, U_2 < z)} \\ &= \lim_{z\to 0} \frac{C(z,z)}{2z - C(z,z)},\end{aligned} \tag{4.11}$$

whereas the upper extremal dependence coefficient is defined as

$$\begin{aligned}
\epsilon_R &= \lim_{z\to 1} P(U_{\text{Max}} > z \,|\, U_{\text{Min}} > z) \\
&= \lim_{z\to 1} \frac{P(U_{\text{Max}} > z, U_{\text{Min}} > z)}{P(U_{\text{Min}} > z)} \\
&= \lim_{z\to 1} \frac{P(U_1 > z, U_2 > z)}{1 - P(U_1 < z, U_2 < z)} \\
&= \lim_{z\to 1} \frac{1 - P(U_1 < z) - P(U_2 < z) + P(U_1 < z, U_2 < z)}{1 - P(U_1 < z, U_2 < z)} \\
&= \lim_{z\to 1} \frac{1 - 2z + C(z,z)}{1 - C(z,z)}. \qquad (4.12)
\end{aligned}$$

Thus the lower extremal dependence coefficient can be interpreted as the probability that the best performer is affected by the worst one provided that the latter has an extremely bad performance, while the upper extremal dependence coefficient measures the probability that the worst performer is affected by the best given that the latter has an extremely good performance.

The following proposition relates the tail dependence coefficients and the extremal dependence coefficients.

Proposition 4.3 Let λ_L and λ_R be the tail dependence coefficients defined by equations (4.9) and (4.10), and ϵ_L and ϵ_R be the corresponding extremal dependence coefficients defined by equations (4.11) and (4.12). Then

$$\epsilon_L = \frac{\lambda_L}{2 - \lambda_L}, \qquad \epsilon_R = \frac{\lambda_R}{2 - \lambda_R}.$$

Proof. From the definition of lower extremal dependence coefficient given in (4.11), and using $\lim_{z\to 0} \frac{C(z,z)}{z} = \lambda_L$ by equation (4.9), we have

$$\begin{aligned}
\epsilon_L &= \lim_{z\to 0} \frac{C(z,z)}{2z - C(z,z)} \\
&= \lim_{z\to 0} \frac{\frac{C(z,z)}{z}}{2 - \frac{C(z,z)}{z}} \\
&= \frac{\lambda_L}{2 - \lambda_L}.
\end{aligned}$$

Similarly, using (4.12) and (4.10),

$$\begin{aligned}\epsilon_R &= \lim_{z\to 1} \frac{1-2z+C(z,z)}{1-C(z,z)} \\ &= \lim_{z\to 1} \frac{\frac{1-2z+C(z,z)}{1-z}}{2-\frac{1-2z+C(z,z)}{1-z}} \\ &= \frac{\lambda_R}{2-\lambda_R}.\end{aligned}$$

□

Table 4.2 summarizes the measures of tail dependency for the copula functions specified in Chapter 3. If the dependency over the right tail is of particular interest to practitioners, then the Gumbel and Pareto copulas should be considered. Mis-specification of the dependence structure, especially the dependency over the tails, may result in devastating consequences, which will be shown in insurance applications in Chapter 6.

Table 4.2: Copulas and Their Measures of Tail Dependency

Copula	**Left tail**			**Right tail**		
	$L(z)$	λ_L	ϵ_L	$R(z)$	λ_R	ϵ_R
Product (independence)	z	0	0	$1-z$	0	0
Clayton	$\frac{(2z^{-\gamma}-1)^{-1/\gamma}}{z}$	$2^{-1/\gamma}$	$\frac{2^{-1/\gamma}}{2-2^{-1/\gamma}}$	$\frac{1-2z+(2z^{-\gamma}-1)^{-1/\gamma}}{1-z}$	0	0
Frank	$\frac{-\frac{1}{\gamma}\ln\left(1+\frac{(e^{-\gamma z}-1)^2}{e^{-\gamma}-1}\right)}{z}$	0	0	$\frac{1-2z-\frac{1}{\gamma}\ln\left(1+\frac{(e^{-\gamma z}-1)^2}{e^{-\gamma}-1}\right)}{1-z}$	0	0
Gumbel	$z^{2^{1/\gamma}-1}$	0	0	$\frac{1-2z+z^{2^{1/\gamma}}}{1-z}$	$2-2^{1/\gamma}$	$2^{1-1/\gamma}-1$
Hougaard	$\frac{2z-1+(1-z)^{2^{1/\gamma}}}{z}$	$2-2^{1/\gamma}$	$2^{1-1/\gamma}-1$	$(1-z)^{2^{1/\gamma}-1}$	0	0
Pareto	$\frac{2z-1+\left(2(1-z)^{-\gamma}-1\right)^{-1/\gamma}}{z}$	0	0	$\frac{\left(2(1-z)^{-\gamma}-1\right)^{-1/\gamma}}{1-z}$	$2^{-1/\gamma}$	$\frac{2^{-1/\gamma}}{2-2^{-1/\gamma}}$

Chapter 5

Risk Measures

Dependent risks are inherent in the business of insurance companies. The prerequisites of managing risk are first understanding risks and then quantifying and measuring risks. The previous chapters of this project are devoted to modeling the dependent risks and identifying the dependence structure. In this chapter, we will incorporate the dependence between risks into risk measure calculations with an emphasis on the tail-based risk measures. This chapter is organized as follows: Section 5.1 provides a review of two widely used tail-based risk measures – value-at-risk (VaR) and conditional tail expectations (CTE). An illustration of calculating tail-based risk measures for functions of dependent risks is given in Section 5.2.

5.1 Introduction

A risk measure π is a mapping from random variable(s) Y to a non-negative real number, i.e., $\pi : Y \longrightarrow \mathbb{R}$. According to Artzner et al. (1999), a function $\pi : Y \longrightarrow \mathbb{R}$ is said to be coherent risk measure for risk Y if it satisfies the following properties:

(i) Monotonicity: For two risks Y_1 and Y_2, if $Y_1 \leq Y_2$, then $\pi(Y_1) \leq \pi(Y_2)$;

(ii) Sub-additivity: For two risks Y_1 and Y_2, $\pi(Y_1 + Y_2) \leq \pi(Y_1) + \pi(Y_2)$;

(iii) Positive homogeneity: If $\alpha > 0$, then $\pi(\alpha Y) = \alpha\pi(Y)$;

(iv) Translation invariance: For all $a \in R$, $\pi(Y + a) = \pi(Y) + a$.

Numerous risk measures have been proposed in insurance and finance literature, for example, value-at-risk (VaR), tail VaR, conditional tail expectation (CTE), conditional VaR, expected shortfalls, Esscher risk measures, and Wang risk measures. Denuit et al. (2005) provided a review of various methods of measuring risks.

For illustrative purposes, we focus on two tail-based risk measures – VaR and CTE, to measure dependent risks.

5.1.1 Value-at-risk (VaR)

The value-at-risk (VaR) summarizes the worst loss with a given level of confidence. Despite its numerous critics, VaR is still one of the most widely used risk measures. VaR has become the benchmark risk measure used by financial analysts and regulators in quantifying the market risk and setting capital requirements for market risk exposures (Denuit et al., 2005).

Definition 5.1 Given a risk Y and a probability level $p \in (0,1)$, the corresponding VaR, denoted by $\mathrm{VaR}(Y,p)$, is defined as

$$P\left(Y \leq \mathrm{VaR}(Y,p)\right) = p.$$

The VaR gives the maximum likely loss at a specified confidence level. If risk Y has a continuous distribution, then $\mathrm{VaR}(Y,p)$ can be defined explicitly with the help of the quantile function F_Y^{-1},

$$\mathrm{VaR}(Y,p) = F_Y^{-1}(p).$$

5.1.2 Conditional tail expectation (CTE)

VaR measures the worst case loss, where the worst case is defined as the upper tail event with $1-p$ probability. One problem with the quantile risk measure is that it does not take into consideration what the loss will be if that $1-p$ worst case event actually occurs. The loss above the quantile and its probabilities do not affect VaR. The conditional tail expectation (CTE) is designed to address such problems with the quantile risk measure.

The conditional tail expectation (CTE) measures the average loss in the worst $100(1-p)\%$ cases, defined as

$$\mathrm{CTE}(p) = E[Y \mid Y > \mathrm{VaR}(Y,p)].$$

5.2 VaR and CTE for the Functions of Dependent Risks

As we have mentioned in Chapter 1, insurers are more concerned about their total risk exposure, which is a function of dependent risks. The function might be a linear form as is the case for the total loss from different lines of business, different securities in a given portfolio, and different geographical locations, or a more complicated form such as reinsurance agreement, which depends on retention and policy limit.

Assume X_1 and X_2 are two dependent risks with joint distribution function $F(x_1, x_2)$ and the associated copula function $C(F_1(x_1), F_2(x_2))$. We are interested in a new risk,

$$Y = g(X_1, X_2), \tag{5.1}$$

where g is a general function form satisfying the condition that its first-order partial derivatives are non-negative.

In order to obtain the distribution function $F_Y(y)$ or its density function $f_Y(y)$ based on the joint distribution function or density function of X_1 and X_2, we can use the method of transformations.

Define two new variables $z_1 = y = g(x_1, x_2)$ and $z_2 = x_2$. z_2 could be other functions of x_1 and x_2 as long as it yields a convenient inverse transformation. For the sake of simplicity, we assume that $z_2 = x_2$ is a simplest function that gives inverse transformation: $x_1 = g^{-1}(z_1, z_2)$ and $x_2 = z_2$. The Jacobian of this transformation is

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ \frac{\partial x_2}{\partial z_1} & \frac{\partial x_2}{\partial z_2} \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial x_1}{\partial z_1} & \frac{\partial x_1}{\partial z_2} \\ 0 & 1 \end{vmatrix} \\ &= \frac{\partial x_1}{\partial z_1}. \end{aligned}$$

Then the joint density of Z_1 and Z_2 is

$$\begin{aligned} f_{Z_1,Z_2}(z_1, z_2) &= f_{X_1,X_2}(g^{-1}(z_1, z_2), z_2)\,|J| \\ &= f_{X_1,X_2}(g^{-1}(z_1, z_2), z_2)\left|\frac{\partial x_1}{\partial z_1}\right| \\ &= f_{X_1,X_2}(g^{-1}(z_1, z_2), z_2)\left|\frac{\partial g^{-1}(z_1, z_2)}{\partial z_1}\right|. \end{aligned} \tag{5.2}$$

The density function $f_Y(y)$ can be obtained by first replacing z_1 and z_2 by y and x_2, respectively, and then integrating the joint density function (5.2),

$$f_Y(y) = \int f_{Y,X_2}(y, x_2)dx_2. \tag{5.3}$$

Even for the simplest function form of $g(X_1, X_2)$, an explicit expression of function in (5.3) can be very hard to derive. Properties of (5.3) can be obtained using numerical evaluation tools such as simulation.

The remainder of this section will concentrate on the linear combinations of dependent risks, with the form

$$Y = \alpha X_1 + \beta X_2. \tag{5.4}$$

In the context of financial and actuarial applications, the new risk Y given in (5.4) can be interpreted as follows. If X_1 and X_2 are rates of returns of two stocks or stock indices, and $\alpha \in [0, 1]$ and $\beta = 1 - \alpha$, then Y is the rate of return of the portfolio. If X_1 and X_2 are losses from two different lines of insurance, and $\alpha = \beta = 1$, then Y is insurer's total loss.

The mean and variance of Y are

$$E[Y] = \alpha E[X_1] + \beta E[X_2],$$

and

$$\text{Var}[Y] = \alpha^2\text{Var}[X_1] + \beta^2\text{Var}[X_2] + 2\alpha\beta\text{Cov}\,(X_1, X_2)\,.$$

If the joint behavior of X_1 and X_2 is modeled by our common random effect model, then the variance of Y can be expressed as

$$\begin{aligned}\text{Var}[Y] &= E_\Theta\Big[\text{Var}[\alpha X_1 + \beta X_2 \,|\, \Theta]\Big] + \text{Var}_\Theta\Big[E[\alpha X_1 + \beta X_2 \,|\, \Theta]\Big] \\ &= E_\Theta\Big[\alpha^2\text{Var}[X_1 \,|\, \Theta] + \beta^2\text{Var}[X_2 \,|\, \Theta]\Big] + \text{Var}_\Theta\Big[\alpha E[X_1 \,|\, \Theta] + \beta E[X_2 \,|\, \Theta]\Big].\end{aligned}$$

If X_1 and X_2 have a copula function C, i.e., $F(x_1, x_2) = C(F_1(x_1), F_2(x_2))$, then the distribution function of Y can be expressed as

$$\begin{aligned}F_Y(y) &= P(Y \le y) \\ &= P(\alpha X_1 + \beta X_2 \le y) \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{\frac{y-\beta x_2}{\alpha}} f(x_1, x_2)dx_1dx_2 \\ &= \int_{-\infty}^{+\infty}\int_{-\infty}^{\frac{y-\beta x_2}{\alpha}} c(F_1(x_1), F_2(x_2))f_1(x_1)f_2(x_2)dx_1dx_2, \end{aligned} \tag{5.5}$$

where $c(F_1(x_1), F_2(x_2))$ is the copula density function defined in (2.9) and (2.10).

The closed form solution for equation (5.5) does not exist in most cases. Numerical methods or simulations are needed to find the statistical properties of Y.

The double integration in equation (5.5) can be reduced to single integration if the joint behavior of X_1 and X_2 is modeled by our common effect model in Chapter 3. In fact, the distribution function of Y, conditional on $\Theta = \theta$, is

$$\begin{aligned}
G(y \,|\, \theta) &= P(\alpha X_1 + \beta X_2 \le y \,|\, \Theta = \theta) \\
&= \int_{-\infty}^{+\infty} P\left(X_2 \le \frac{y - \alpha x_1}{\beta} \,\middle|\, X_1 = x_1, \Theta = \theta \right) f_1(x_1) dx_1 \\
&= \int_{-\infty}^{+\infty} \left[H_2\left(\frac{y - \alpha x_1}{\beta} \right) \right]^{\theta} d\varphi\left(-\ln H_1(x_1) \right),
\end{aligned}$$

where φ is the Laplace transform of the common random effect Θ. As a result, the unconditional distribution function of Y can be expressed as

$$F_Y(y) = \int_{-\infty}^{+\infty} G(y \,|\, \theta) f(\theta) d\theta,$$

where $f(\theta)$ is the probability density function of Θ.

The explicit expressions of $F_Y(y)$ for the Pareto copula and the Gumbel copula are shown in the following sections.

5.2.1 Pareto copula

The density function of Pareto copula can be derived by taking derivatives of equation (3.13), namely

$$\begin{aligned}
c(u_1, u_2) &= \frac{\partial^2 C(u_1, u_2)}{\partial u_1 \partial u_2} \\
&= (1+\gamma)\Big[(1-u_1)^{-\gamma} + (1-u_2)^{-\gamma} - 1 \Big]^{-2-\frac{1}{\gamma}} (1-u_1)^{-1-\gamma} (1-u_2)^{-1-\gamma}.
\end{aligned}$$

Therefore, the distribution function of Y given by equation (5.5) can be written as

$$\begin{aligned}
F_Y(y) = \int_{-\infty}^{+\infty} \int_{-\infty}^{\frac{y-\beta x_2}{\alpha}} & (1+\gamma)\Big[(1-F_1(x_1))^{-\gamma} + (1-F_2(x_2))^{-\gamma} - 1 \Big]^{-2-\frac{1}{\gamma}} \\
& \times \big[1 - F_1(x_1)\big]^{-1-\gamma} \big[1 - F_2(x_2)\big]^{-1-\gamma} f_1(x_1) f_2(x_2) dx_1 dx_2,
\end{aligned}$$

where $F_i(x_i)$ and $f_i(x_i)$, $i = 1, 2$, are the univariate marginal distribution function and density function, respectively.

5.2.2 Gumbel copula

The density function of Gumbel copula can be obtained by taking derivatives of equation (3.6), as follows

$$
\begin{aligned}
c(u_1,u_2) =& \frac{\partial^2 C(u_1,u_2)}{\partial u_1 \partial u_2} \\
=& \frac{e^{-\left[(-\ln u_1)^\gamma + (-\ln u_2)^\gamma\right]^{\frac{1}{\gamma}}}}{u_1 u_2} \left[(-\ln u_1)^\gamma + (-\ln u_2)^\gamma\right]^{-2+\frac{2}{\gamma}} \\
& \times \left[(\ln u_1)(\ln u_2)\right]^{\gamma-1} \left\{1 + (\gamma-1)\left[(-\ln u_1)^\gamma + (-\ln u_2)^\gamma\right]^{-\frac{1}{\gamma}}\right\}. \qquad (5.6)
\end{aligned}
$$

Therefore, the distribution function of Y given by equation (5.5) can be expressed as

$$
\begin{aligned}
F_Y(y) =& \int_{-\infty}^{+\infty} \int_{-\infty}^{\frac{y-\beta x_2}{\alpha}} \frac{e^{-\left[(-\ln F_1(x_1))^\gamma + (-\ln F_2(x_2))^\gamma\right]^{\frac{1}{\gamma}}}}{F_1(x_1)F_2(x_2)} \\
& \times \left[(-\ln F_1(x_1))^\gamma + (-\ln F_2(x_2))^\gamma\right]^{-2+\frac{2}{\gamma}} \left[(\ln F_1(x_1))(\ln F_2(x_2))\right]^{\gamma-1} \\
& \times \left\{1 + (\gamma-1)\left[(-\ln F_1(x_1))^\gamma + (-\ln F_2(x_2))^\gamma\right]^{-\frac{1}{\gamma}}\right\} f_1(x_1) f_2(x_2) dx_1 dx_2, \qquad (5.7)
\end{aligned}
$$

where $F_i(x_i)$ and $f_i(x_i)$, $i = 1, 2$, are the univariate marginal distribution function and density function, respectively.

Chapter 6

Applications

In this chapter, the statistical application of the copula modeling approach to insurance data is discussed. The joint behavior of losses and loss adjustment expenses in insurance claims (data) are investigated. The insurance applications based on the fitted model are illustrated.

6.1 Data and Previous Work

The insurance loss data set was supplied by the Insurance Services Office (ISO) and consists of liability claims of an insurance company. This data set was available from various sources, including the R package, “Copula”, and the personal webpage of Professor Edward W. (Jed) Frees[1].

This data set contains 1500 randomly selected claims. For each claim, the indemnity payment (loss), the allocated loss adjustment expense (ALAE), and the policy limit were recorded. 34 claims that had indemnity payments greater than the policy limit were censored. A statistical summary of the data is shown in Table 6.1.

Figure 6.1 shows the scatterplot of losses versus expenses. These plots suggest a positive dependence between the loss and ALAE, and the dependence appears to become stronger at high values of losses.

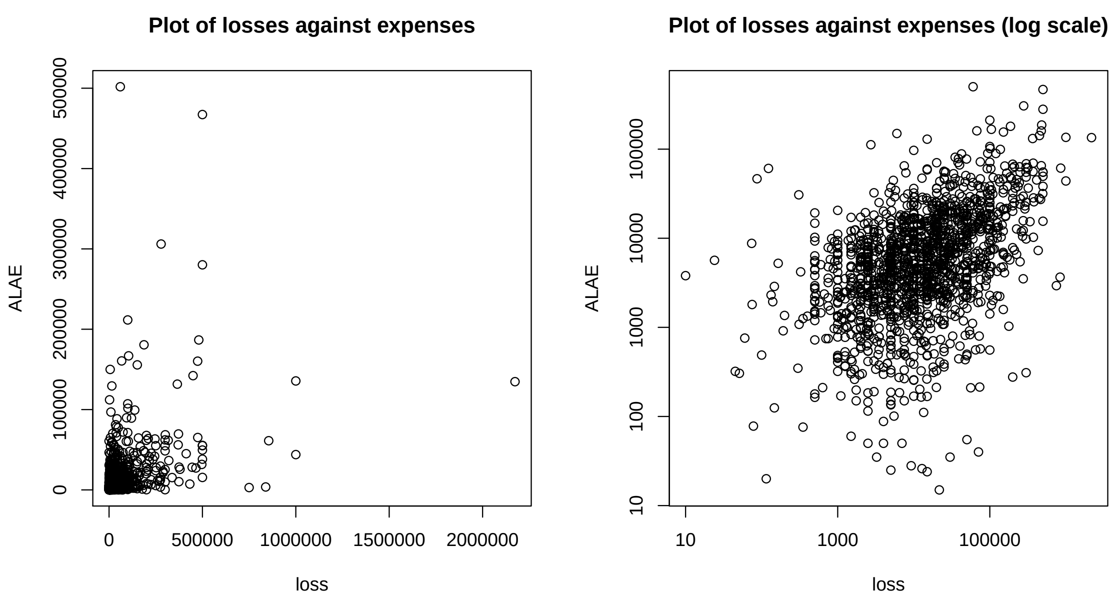
Klugman and Parsa (1999) fitted an inverse paralogistic and an inverse Burr distribution to the loss and ALAE data, respectively, and then used a Frank copula to model the joint

[1]http://research3.bus.wisc.edu/file.php/129/DataCode/LOSSDATA.txt

Table 6.1: Statistical Summary of Losses and Expenses Data

	Loss	ALAE	Policy Limit	Loss (Uncensored)	Loss (Censored)
Number	1,500	1,500	1,352	1,466	34
Average	41,208	12,588	559,098	37,110	217,941
Standard Deviation	102,748	28,146	418,649	92,513	258,205
Minimum	10	15	5,000	10	5,000
25 Percentile	4,000	2,333	300,000	3,750	50,000
Median	12,000	5,471	500,000	11,048	100,000
75 Percentile	35,000	12,572	1,000,000	32,000	300,000
Maximum	2,173,595	501,863	7,500,000	2,173,595	1,000,000

Figure 6.1: Scatterplots of Loss against ALAE



distribution of losses and expenses. Frees and Valdez (1998) chose Pareto distributions for the univariate marginals, and used Q-Q plots and the Akaike information criterion (AIC) for the model selection among the Clayton, Frank, and Gumbel copulas. Both Q-Q plots and AIC suggested that the Gumbel copula is preferred.

Frees and Valdez (1998) also used the estimated bivariate distribution of losses and expenses to calculate reinsurance premiums and estimate expenses for pre-specified losses. Simulations were performed to estimate reinsurance premiums based on a pro-rata sharing of expenses. If the unrealistic assumption of independence between losses and expenses is made, reinsurance premiums would be substantially undervalued for higher policy limits and higher retention values set by the reinsured.

Another empirical investigation using this data set was by Denuit et al. (2005). They used the losses and expenses data as a case study for modeling Archimedean copulas. Denuit et al. (2005) confirmed that the Gumbel copula provides the best fit to the data, and the Frank copula also gives a very good fit.

6.2 Fitting Copula Models

To fit a copula to losses and expenses data, we need to determine the appropriate marginal distributions first, then choose the function form of copula. A variety of methods can be used for copula selection, including, among others, the visual detection from the empirical distributions, log-likelihood values, Akaike Information Criterion (AIC), and Bayesian Information Criterion (BIC).

6.2.1 Fitting marginal distributions

The first step of copula model fitting is to determine the appropriate marginal distributions. We present the fit of univariate marginals with generalized Pareto distribution. With location parameter, μ, scale parameter, θ, and shape parameter, γ, the distribution function of the generalized Pareto distribution is

$$F(x) = 1 - \left(1 + \gamma \frac{x-\mu}{\theta}\right)^{-\frac{1}{\gamma}}.$$

Our choice of generalized Pareto distributions for modeling univariate marginals is based on two reasons. Firstly, the generalized Pareto distributions would improve the overall fit as it has one more parameter. The second and more important reason is that the generalized Pareto model would be more flexible than the 2-parameter Pareto distribution proposed in Frees and Valdez (1998) and Denuit et al. (2005). If one is interested in the losses above a high threshold, then the generalized Pareto model can deal with the threshold excesses easily by setting the location parameter equal to the threshold.

Since the data set contains censored losses, the log-likelihood function is given by

$$\ln L(\mu, \theta, \gamma) = \sum_{i=1}^{n}(1-\delta_i)\ln f(x_i) + \sum_{i=1}^{n}\delta_i \ln(1-F(x_i)),$$

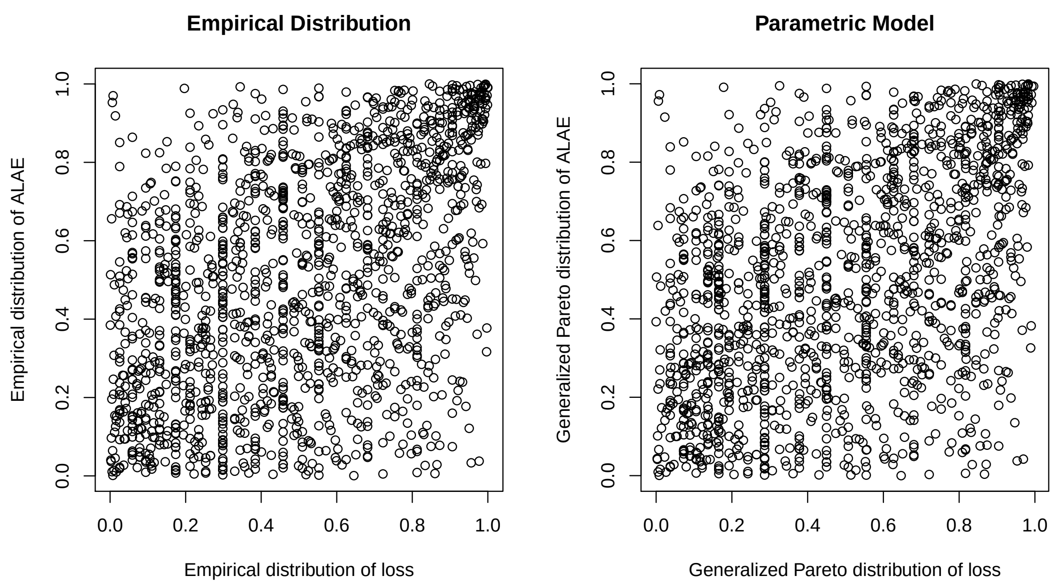
where δ_i is the censoring indicator, with $\delta_i = 0$ indicating uncensored case and $\delta_i = 1$ indicating censored case.

Table 6.2: Fitting Marginal Distributions

Parameter	**Loss**		**ALAE**	
	Estimate	Standard Error	Estimate	Standard Error
Location μ	10	8.472	15	4.518
Scale θ	12,692.9472	612.485	6,773.2501	289.203
Shape γ	0.8834	0.051	0.4529	0.036
Log-likelihood	-16,536.176		-15,410.135	

Table 6.2 summarizes the results from the maximum likelihood estimation fitting of the marginal distributions. The maximum likelihood estimates of the locations parameters are the minimum values in the sample, and their standard errors are based on their order statistics. The overall fit shows some minor improvement over the Pareto marginal distributions in Frees and Valdez (1998) and Denuit et al. (2005), evidenced by smaller AIC values. Since our generalized Pareto model has three parameters, while the parametric model chosen by Frees and Valdez (1998) and Denuit et al. (2005) has two, we compute and compare the AIC values for each model. AIC of the generalized Pareto model for loss is $2k - 2\ln L = 2(3) + 2(16,536.176) = 33,078.35$, which is smaller than the AIC for the 2-parameter Pareto model of loss, $2(2) + 2(16,537.369) = 33,078.74$. The generalized Pareto estimation of ALAE also has an improvement, with the AIC of $2(3)+2(15,410.135) = 30,826.27$, against the AIC value of $2(2)+2(15,413.449) = 30,830.90$

Figure 6.2: Scatterplots of Empirical and Marginal Distributions



for the 2-parameter Pareto model. However, the comparison of the BIC values, which penalizes extra parameters more strongly than AIC does, shows that the 2-parameter Pareto model is preferable.

6.2.2 Visualizing dependence structure

Before fitting copula models, we first look at the joint behavior of the empirical and marginal distributions of loss and ALAE data. A good model of the bivariate distribution has statistical properties that resemble those of the empirical distributions. Figure 6.2 gives the scatterplots of the empirical and marginal distributions of losses versus those of ALAE; the empirical distribution is given by

$$\widehat{F}(s) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}\{x_i \leq s\},$$

where n is the number of observations, and $\mathbb{1}$ is the indicator function.

Marginal distributions are from the fitted generalized Pareto models in the previous section. The joint behavior of marginal distributions is similar to that of the empirical distribution, which once again justifies the choice of the generalized Pareto marginal distributions. Both plots show strong right tail dependence and relatively weak left tail dependence, suggesting Gumbel and Pareto copulas could be good choices for the joint behavior of losses and expenses.

6.2.3 Fitting copula models

The next step of the estimation process is to feed the marginal distributions obtained from the previous step to copula functions to estimate the dependence parameter. The maximum likelihood estimator is obtained by maximizing

$$\ln L(\gamma) = \sum_{i=1}^{n} \ln c(\hat{u}_{1i}, \hat{u}_{2i}; \gamma),$$

with respect to the dependence parameter γ, where c denotes the copula density given by (2.9), and $\hat{u_1}$ and $\hat{u_2}$ are marginal distributions from the previous step. The copula functions to be fitted include all the copula expressions derived through modeling the distribution functions and the survival functions, as listed in Table 3.2.

Maximum likelihood estimates of selected copula functions are shown in Table 6.3. Each copula model is estimated based on empirical marginal distributions and parametric marginal distributions. The Gumbel copula has the largest log-likelihood value, and therefore produces the best fit for both empirical and parametric marginal distributions. The Pareto copula also produces a very good fit. We recall that the Gumbel copula is generated by modeling the distribution functions and assuming that the common random effect follows a positive stable distribution, whereas the Pareto copula comes from modeling the survival functions and assuming that the common random effect has a gamma distribution. Insurance applications in the next two sections are based on the fitted Gumbel and Pareto copulas.

Perspective plots of the fitted copulas and their implied bivariate density functions are shown in Figure 6.3 and Figure 6.4, respectively. Both Gumbel and Pareto copulas have strong right tail dependence and relatively weak left tail dependence. The trivial difference between these two copulas is that the Gumbel copula has a slightly heavier left tail than the Pareto copula. Given the relationships between the marginal distributions as shown in Figure 6.2, it makes perfect sense why the Gumbel copula gives the best fit and the Pareto

Table 6.3: Maximum Likelihood Estimates of the Copula Functions

Copula	**Empirical Distribution**		
	Dependence parameter	Standard error	Log-likelihood value
Clayton	0.5196	0.0425	93.833
Frank	3.1014	0.1680	172.506
Gumbel	1.4432	0.0288	206.995
Hougaard	1.3773	0.0277	138.372
Pareto	0.7752	0.0459	201.662
Copula	**Parametric Marginal Distribution**		
	Dependence parameter	Standard error	Log-likelihood value
Clayton	0.5137	0.0436	86.640
Frank	3.1484	0.1699	172.885
Gumbel	1.4555	0.0295	203.774
Hougaard	1.3788	0.0282	132.701
Pareto	0.7972	0.0471	198.691

Figure 6.3: Perspective Plots of the Fitted Copula Models

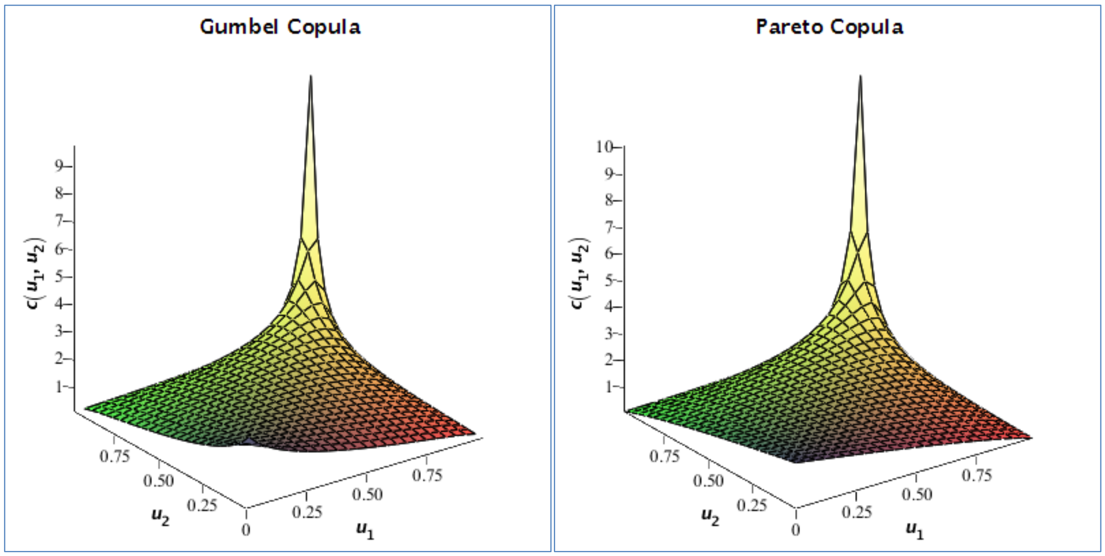
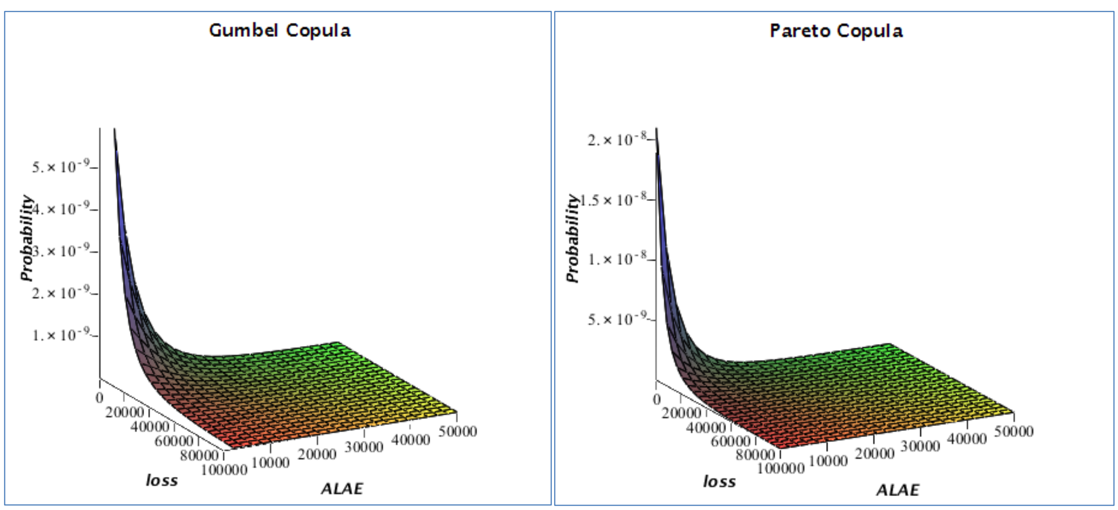


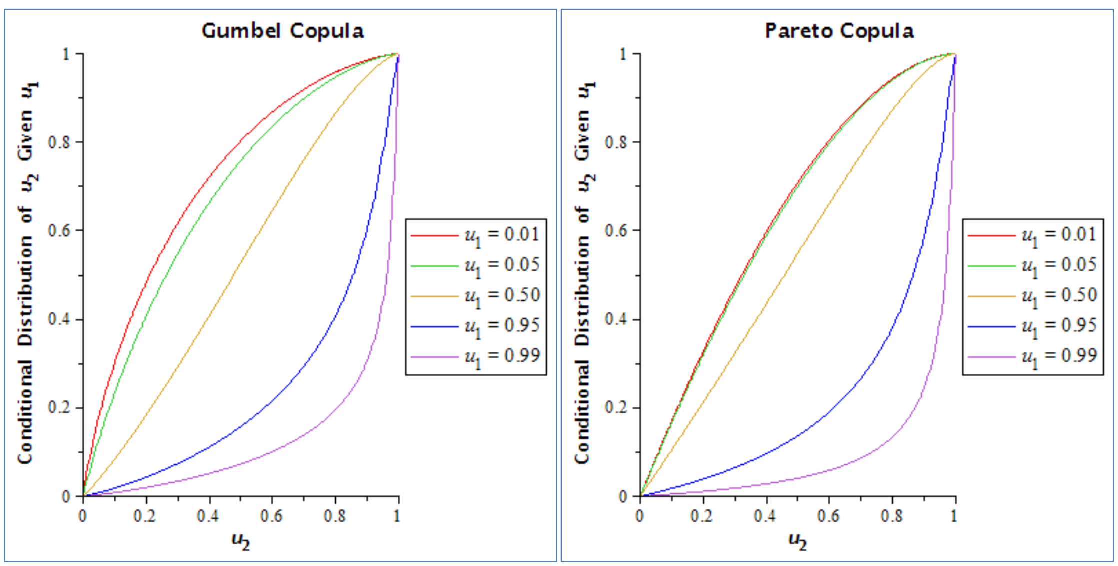
Figure 6.4: Joint Probability Density Functions



copula is in the second place. That is why we emphasize the importance of visual detection of the dependence structure before fitting copula models. A final comment regarding fitting copula to losses and expenses data is that the Frank copula, which was chosen by Klugman and Parsa (1999), should never be preferred, because the Frank copula is symmetric and cannot capture the strong dependence in the right tail only.

Figure 6.5 gives the conditional distributions based on the fitted copulas. For the purpose of comparison, the quantiles of 0.01, 0.05, 0.50, 0.95, and 0.99 are selected. As far as high quantiles (right tail) are concerned, results from the two fitted copulas are very close. However, over the left tail, the dependence in the Pareto copula is quite weak compared to that in the Gumbel copula.

Figure 6.5: Conditional Distribution Functions Based on the Fitted Copulas



Measures of dependence can be computed from the estimated dependence parameters given in Table 4.1. For example, the dependence parameter of 1.4555 in the fitted Gumbel copula corresponds to Kendall's tau correlation measure of 0.313. Table 6.4 gives the rank correlations based on the fitted copula models. The fitted Gumbel copula has rank correlations very close to those directly estimated from the raw data, which once again shows that the Gumbel copula gives the best fit to the losses and expenses data.

Tail dependency measures based on our fitted copula models are presented in Figure 6.6 and Table 6.5. The upper and lower limits of tail concentration functions as shown in Figure 6.6 correspond to the tail dependence coefficients in Table 6.5. Both the tail dependence coefficients and extremal dependence coefficients indicate that losses and expenses are asymptotically independent in the left tails, and asymptotically dependent in the right tails. As the insurance loss approaches its maximum loss amount, there is a probability of around 0.40 that the loss adjustment expense also reaches its maximum amount.

Table 6.4: Measures of Dependence Based on the Fitted Copulas

	Spearman's Rho	**Kendall's Tau**
Raw Data	0.452	0.315
Gumbel Copula	0.448	0.313
Pareto Copula	0.414	0.285

Table 6.5: Measures of Tail Dependency Based on the Fitted Copulas

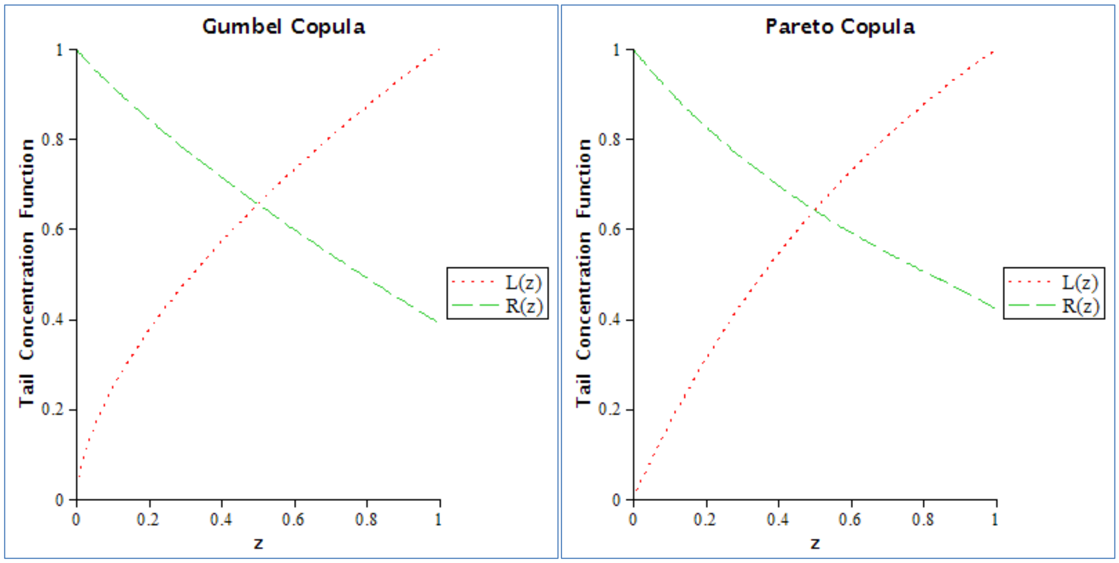
	Tail Dependence Coefficients		**Extremal Dependence Coefficients**	
	Left Tail	Right Tail	Left Tail	Right Tail
Gumbel Copula	0	0.390	0	0.242
Pareto Copula	0	0.419	0	0.265

6.3 Tail-based Risk Measures for Total Cost of Claim

After estimating and selecting the bivariate models for dependent risks, now we can take a further step to quantify the impact of dependency between risks. In this and next sections, we aim at answering two questions: how does ignorance or mis-specification of dependency affect risk measures? If the unrealistic assumption of independence is made, what is the magnitude of insurance mispricing?

To answer the first question, we consider the sum of two dependent risks that has been discussed in Chapter 5 as an illustration. Given the bivariate distribution of losses and

Figure 6.6: Tail Concentration Functions Based on the Fitted Copulas



expenses, we are interested in the total cost of claim, which is equal to the sum of losses and expenses. Numerical integration can be used to find the quantitative properties of interest. Simulation, however, is much simpler.

The marginal distributions of losses and expenses and the dependence structure captured by copulas are important inputs into the simulation algorithm. To estimate VaR and CTE, we simulate 500,000 observations of losses and expenses using the estimated parameters of the marginal distributions and the dependence parameter that specifies copula. We add up the simulated losses and loss adjustment expenses to get the total cost for each claim. The VaR can be obtained from the quantiles of the distribution of the total cost, and the CTE is calculated as the mean of the simulated value above its corresponding quantile.

The results of VaR and CTE for the Gumbel, Pareto, and Frank copulas, and the independence case are presented in Table 6.6 and Table 6.7. For comparison purposes, four quantiles – 90%, 95%, 97.5%, and 99%, were selected.

If independence is assumed, VaR is understated at all four chosen quantiles. The magnitude of underestimation ranges from 1% at the 90th percentile, to around 6% at the 95th and 97.5th percentiles. However, if the dependence structure is mistakenly specified as the

Frank copula, then the VaR at 90th and 95th percentiles overestimates those based on the Gumbel or Pareto copula. But at the 97.5th and 99th percentiles, the Frank copula gives an underestimation of VaR.

As for the CTE estimates, mis-specification of the dependence structure and ignorance of dependency make no difference. Both result in underestimation of the average cost in the worst case scenarios, ranging from about 2% as compared with the Gumbel copula to around 4% as compared with the Pareto copula.

These results confirm the importance of selecting an appropriate dependence structure in calculating risk measures. As expected, the unrealistic assumption of independence tends to understate the actual risk measured by the VaR and CTE. But mis-specification of dependence structure may lead to an overestimation or underestimation of the VaR. This suggests that mis-specification of dependency may do as much harm as, if not more than, the assumption of independence.

Table 6.6: Comparison of Simulation-based Value-at-Risk Estimates

Copula	**Quantile**			
	90%	95%	97.5%	99%
Gumbel	119,430	223,255	409,120	901,875
Pareto	119,402	222,856	412,470	933,255
Independence	118,435	212,587	385,426	849,477
Frank	125,337	225,528	399,816	877,036

6.4 Pricing Reinsurance Contracts

Knowing the joint distribution of losses (X_1) and expenses (X_2) also allows us to estimate reinsurer's expected payment under a reinsurance agreement such as the one discussed in Frees and Valdez (1998). Suppose there is a reinsurance policy with limit L and insurer's retention R. Also, assume that the reinsurer pays a pro-rata share of expenses, which is $\frac{X_1-R}{X_1}$ for losses below the policy limit and $\frac{L-R}{L}$ for losses equal or above the policy limit. The reinsurer's payment is

Table 6.7: Comparison of Simulation-based CTE Estimates

Copula	Quantile			
	90%	95%	97.5%	99%
Gumbel	714,740	1,268,990	2,241,683	4,722,434
Pareto	723,319	1,286,481	2,275,776	4,795,396
Independence	692,263	1,228,894	2,176,917	4,613,115
Frank	700,704	1,236,247	2,177,815	4,586,743

$$g(X_1, X_2) = \begin{cases} 0, & \text{if } X_1 < R; \\ X_1 - R + \frac{X_1 - R}{X_1} X_2, & \text{if } R \leq X_1 < L; \\ L - R + \frac{L-R}{L} X_2, & \text{if } X_1 \geq L. \end{cases}$$

The reinsurance premium can be calculated as $E[g(X_1, X_2)]$. Simulation-based reinsurance premiums for independence, Gumbel copula, and Pareto copula are presented in Tables 6.8, 6.9, and 6.10, respectively. Premiums are calculated using the 500,000 simulations for each specification of dependence structure as presented in Section 6.3.

For all three cases, reinsurance premiums decrease as insurers' retention increases. This makes perfect sense because when the reinsured retains larger amount of loss, reinsurer's expected payment falls, and as a result, reinsurance premiums decrease. For a given ratio of insurers' retention to policy limit, an increase in policy limit may lead to increase or decrease in reinsurance premiums. That's because two forces are working in opposite directions when policy limit increases. On the one hand, increase in policy limit means that insurers cede more losses and expenses to reinsurer, which tends to increase reinsurer's expected payment. On the other hand, insurers' retention also increases because of the constant ratio of insurers' retention to policy limit, which means that reinsurer's expected payment will decrease. The total effect depends on the ratio of insurers' retention to policy limit. For example, if the retention is zero, then the reinsurance contract is the same as regular insurance policy, reinsurance premiums always increase as policy limit increases. If the ratio of insurers' retention to policy limit equals 0.25 or 0.50, reinsurance premiums first increase then decrease. But at a ratio of 0.75 or 0.95, reinsurance premiums always decrease as policy limit increases.

If the unrealistic assumption of independence between losses and expenses is made, mispricing of insurance contracts would result. Table 6.11 gives the ratios of dependence to independence reinsurance premiums based on the fitted models. A ratio above 1.0 suggests an undervaluation of reinsurance contract under the assumption of independence. An increase in the ratio of insurers' retention to policy limit leads to an increase in the magnitude of mispricing. As the policy limit increases, the ratios of dependence to independence reinsurance premiums tend to rise first, then fall after the policy limit reaches high percentiles. These results underscore the importance of selecting the appropriate model for extremal dependence.

Table 6.8: Simulation-based Reinsurance Premiums – Independence Case

Policy Limit (L)	**Ratio of Insurer's Retention to Policy Limit (R/L)**				
	0.00	0.25	0.50	0.75	0.95
10,000	19,758	11,926	7,126	3,285	623
100,000	38,484	14,523	7,392	3,088	553
500,000	53,417	13,231	6,435	2,620	460
1,000,000	59,241	12,297	5,901	2,384	419

Table 6.9: Simulation-based Reinsurance Premiums – Gumbel Copula

Policy Limit (L)	**Ratio of Insurer's Retention to Policy Limit (R/L)**				
	0.00	0.25	0.50	0.75	0.95
10,000	19,763	13,311	8,319	3,956	765
100,000	38,513	17,114	9,030	3,863	702
500,000	53,515	15,027	7,396	3,053	542
1,000,000	59,387	13,647	6,630	2,710	481

Table 6.10: Simulation-based Reinsurance Premiums – Pareto Copula

Policy Limit (L)	Ratio of Insurer's Retention to Policy Limit (R/L)				
	0.00	0.25	0.50	0.75	0.95
10,000	19,745	13,256	8,295	3,952	765
100,000	38,508	17,286	9,147	3,922	714
500,000	53,500	15,167	7,506	3,104	553
1,000,000	59,469	13,889	6,796	2,804	498

Table 6.11: Ratios of Dependence to Independence Reinsurance Premiums

Policy Limit (L)	Ratio of Insurer's Retention to Policy Limit (R/L)									
	0.00		**0.25**		**0.50**		**0.75**		**0.95**	
	Gumbel	Pareto	Gumbel	Pareto	Gumbel	Pareto	Gumbel	Pareto	Gumbel	Pareto
10,000	1.000	0.999	1.116	1.112	1.167	1.164	1.204	1.203	1.227	1.227
100,000	1.001	1.001	1.178	1.190	1.222	1.237	1.251	1.270	1.270	1.291
500,000	1.002	1.002	1.136	1.146	1.149	1.166	1.165	1.185	1.178	1.203
1,000,000	1.002	1.004	1.110	1.129	1.124	1.152	1.137	1.176	1.147	1.188

Chapter 7

Concluding Remarks

In this project, the joint behavior of two random variables is studied using models of common random effects. Following Oakes (1989, 1994) and Marshall and Olkin (1988), dependency between two random variables is modeled through common random effects. Bivariate distribution and survival functions are generated with univariate marginals as parameters, which greatly simplifies the construction of copulas. Commonly used copulas, such as the Clayton, Frank, and Gumbel copulas, can be generated using common random effects. Measures of tail dependency are applied for the copula model selections. Tail-based risk measures for the functions of two dependent variables are investigated for particular interests.

Our contributions made in this research project can be described as follows. Firstly, a unified approach is proposed to study the dependency between random variables. Oakes (1989, 1994) applied the frailty model to account for the dependencies among multiple lives. Marshall and Olkin (1988) illustrated the use of mixture models to construct multivariate distributions. We combine their methods together and use models of common random effects to study both bivariate distributions and survival functions. The second contribution is the use of measures of tail dependency for copula model selection. The conventional tools for model selection such as AIC or BIC focus on the overall fit to the data, and as a result the selected copula model may or may not be able to capture the dependency in the tails. Finally, risk measures of functions of dependent risks are investigated. We incorporate the dependency between random variables into the calculation of tail-based risk measures. The financial consequences of mis-specification of dependency and ignorance of dependence are illustrated using insurance losses and expenses data.

This work can be further extended and continued in many ways. Several distributions of the common random effect Θ are illustrated in this project. The modeling framework can be applied to other distributions of Θ. For example, if the common random effect has Laplace transform $\varphi(s) = (1-\gamma)/(e^s - \gamma)$, then the resulting copula belongs to the Ali-Mikhail-Haq family (Nelsen, 2006). Then, more interesting results, including the behavior of tail dependency, could be derived from other distributions of the common random effect. Secondly, more work can be done on the risk measures of functions of dependent risks. We tried the linear combinations of dependent risks in this project, and used simulations to find the statistical quantities of interest. Approximations of distribution functions of the functions of dependent risks such as bounds may be explored to gain more insight into the impact of dependency between risks.

Appendix A

Fitting Copulas to Data

Sklar's Theorem suggests that the construction of a model for the joint behavior of m random variables $X_1, X_2, ..., X_m$ can be broken into two parts: the estimation of the marginal distribution functions, $F_1, F_2, ..., F_m$, and the estimation of the dependence parameter(s) in copula C.

A.1 Forming a Pseudo-sample for the Copula

Let $\widehat{F}_1, \hat{F}_2, \ldots, \widehat{F}_m$ denote estimates of the marginal distribution functions. The pseudo-sample from the copula consists of the vector $\widehat{U}_1, \widehat{U}_2, \ldots, \widehat{U}_m$, where

$$\widehat{U}_t = (\widehat{U}_{t,1}, \ldots, \widehat{U}_{t,m})' = (\widehat{F}_1(X_{t,1}), \ldots, \widehat{F}_m(X_{t,m}))', \quad t = 1, 2, ..., n.$$

Possible methods of obtaining the marginal estimates $F_1, F_2, \ldots, F_m$ include the following:

(1) **Parametric estimation**

We can choose an appropriate parametric model for the data to get $\widehat{F}_1(X_{t,1}), \ldots, \widehat{F}_m(X_{t,m})$.

(2) **Non-parametric estimation**

We could estimate the empirical distribution function $\widehat{F}_i$ from $X_{1,i}, X_{2,i}, \ldots, X_{n,i}$ by using

$$\widehat{F}_i(x) = \frac{1}{n} \sum_{t=1}^{n} \mathbb{1}\{X_{t,i} \leq x\}, \quad i = 1, 2, ..., m,$$

where n is the number of observations, and $\mathbb{1}$ is the indicator function.

(3) **Extreme value theory for the tails**

If one is interested in the distribution in the tails, for example, insurance losses above a threshold, then parametric models that provide good overall fits to the data may not be useful. Empirical distribution functions are also poor estimators of the underlining distribution in the tails. Extreme value theory can be used to fit a generalized Pareto distribution for the tails.

A.2 Maximum Likelihood Estimation

Let C_γ denote a parametric copula, where γ is the dependence parameter(s) to be estimated. The maximum likelihood estimator can be obtained by maximizing

$$\ln L(\widehat{U}_1, \dots, \widehat{U}_m; \gamma) = \sum_{t=1}^{n} \ln c(\widehat{U}_t)$$

with respect to γ, where c is the copula density function with dependence parameter γ, defined as

$$c(u_1, u_2, \dots, u_m) = \frac{\partial^m C_\gamma(u_1, u_2, \dots, u_m)}{\partial u_1 \partial u_2 \dots \partial u_m},$$

and $\widehat{U}_t$ denotes the t-th pseudo-observation from the copula.

A.3 Estimation Based on Rank Correlations

Suppose that the assumed model is of the form $F(x_1, x_2) = C(F_1(x_1), F_2(x_2); \gamma)$, where γ is the dependence parameter to be estimated. For many copulas, a functional relationship exists between either Kendall's tau and γ or Spearman's rho and γ (Table 4.1). For example, if we have a relationship of the form $\tau_K = g(\gamma)$, then the chosen copula is calibrated by $\hat{\gamma} = g^{-1}(\tau_K)$.

A.4 Full Maximum Likelihood

Alternatively, we can estimate all parameters using the full maximum likelihood approach. Let $C(x_1, x_2; \gamma)$ be a bivariate copula model with dependence parameter γ. Assume that both C and F_i are differentiable. The joint density is

$$f(x_1, x_2) = c(F_1(x_1 \,|\, \beta_1), F_2(x_2 \,|\, \beta_2); \gamma) f_1(x_1 \,|\, \beta_1) f_2(x_2 \,|\, \beta_2),$$

where c is the copula density defined as

$$c(F_1(x_1 \,|\, \beta_1), F_2(x_2 \,|\, \beta_2)) = \frac{\partial^2 C(F_1(x_1 \,|\, \beta_1), F_2(x_2 \,|\, \beta_2); \gamma)}{\partial F_1(x_1 \,|\, \beta_1) \partial F_2(x_2 \,|\, \beta_2)},$$

and $f_i(x_i \,|\, \beta_i)$ is the density function corresponding to $F_i(x_i \,|\, \beta_i)$, and β_i's are parameter(s) for the marginal distributions of X_i.

The full maximum likelihood estimator can be obtained by maximizing the log-likelihood function

$$\ln L(x_1, x_2; \beta_1, \beta_2, \gamma) = \sum_{j=1}^{n} \ln c(F_1(x_{1j} \,|\, \beta_1), F_2(x_{2j} \,|\, \beta_2)); \gamma) + \sum_{i=1}^{2} \sum_{j=1}^{n} \ln f_i(x_{ij} \,|\, \beta_i)$$

with respect to β_1, β_2, and γ.

In this project, we use the two-step maximum likelihood method. Firstly, marginal distributions for losses and expenses are estimated. Marginal distribution functions are then fed to the copula functions to estimate the dependence parameter.

Appendix B

Simulating Copulas and Bivariate Distributions

Simulation is a powerful numerical evaluation approach that can be used to gain insight into the behavior of dependent risks. In this project, simulations are used on at least two occasions. Simulations help to visualize the dependence property, especially the tail dependence structure in Chapter 4. In Chapters 5 and 6, the closed form solution for the distribution of the functions of dependent risks is not available, where simulations are applied to generate the tail-based risk measures. This section outlines the procedures used in simulating copulas and bivariate distributions.

B.1 Conditional Sampling

Conditional sampling is a simple method for generating random variables from a known copula function. The theoretical basis of this approach is Proposition 2.1. The conditional distribution of X_2 given $X_1 = x_1$ is given by $F_{2\,|\,1}(x_2\,|\,x_1) = C_1(u_1, u_2)$. If C_1 is invertable algebraically, then X_2 can be simulated by the conditional distribution. The steps of simulating copulas by conditioning are as follows:

(i) Draw two independent uniformly distributed variables (v_1, v_2) from $[0, 1]$.

(ii) Set $u_1 = v_1$.

(iii) Generate x_1 by inverting the marginal distribution function, $x_1 = F^{-1}(u_1)$.

(iv) Invert the conditional distribution $C_1(u_1, u_2)$ and get $u_2 = C_1^{-1}(v_2 \,|\, u_1)$.

(v) Generate x_2 by inverting the marginal distribution function, $x_2 = F^{-1}(u_2)$.

For example, the conditional distribution of the Pareto copula has the following form:

$$C_1(u_1, u_2) = 1 - \left[(1-u_1)^{-\gamma} + (1-u_2)^{-\gamma} - 1\right]^{-1-\frac{1}{\gamma}}(1-u_1)^{-1-\gamma},$$

and u_2 can be solved in closed form as

$$u_2 = 1 - \left[1 - (1-u_1)^{-\gamma} + (1-u_1)^{-\gamma}(1-v_2)^{-\frac{\gamma}{1+\gamma}}\right]^{-\frac{1}{\gamma}}.$$

B.2 Sampling by Mixture

The conditional distribution of Gumbel copula cannot be inverted algebraically. To generate random variables from the Gumbel copula using conditional sampling, we have to calculate $u_2 = C_1^{-1}(v_2 \,|\, u_1)$ iteratively. Marshall and Olkin (1988) proposed a simulating approach based on mixtures of powers. The following steps show how this algorithm can be used to generate random variables as an alternative to the conditional sampling:

(i) Draw a random variable θ with Laplace transform $\varphi(s)$.

(ii) Draw two independent uniformly distributed variables (v_1, v_2) from $[0, 1]$.

(iii) Set $u_i = \varphi(-\theta^{-1} \ln v_i)$ for $i = 1, 2$.

(iv) Generate x_i by inverting the marginal distribution functions, $x_i = F^{-1}(u_i)$ for $i = 1, 2$.

For example, the Gumbel copula can be simulated by first drawing θ with Laplace transform $\varphi(s) = e^{-s^{1/\gamma}}$ and two independent uniform variables (v_1, v_2), then generating $u_i = e^{-(-\frac{1}{\theta}\ln v_i)^{1/\gamma}}$ for $i = 1, 2$.

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